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Matrices aléatoires: grande dimension et résolution exacte

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“La victoire appartient au plus opiniâtre.”

— Roland Garros

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Résumé. Cette thèse explore certains aspects de la solvabilité exacte en théorie des matrices aléatoires. Elle est structurée en deux parties principales.

La première partie traite d'un problème en grande dimension, sur comportement asymptotique des polynômes caractéristiques de matrices aléatoires. Nous nous concentrons sur deux modèles intégrables. Le premier est l'Ensemble de Ginibre Elliptique, une interpolation gaussienne entre l'Ensemble de Ginibre et son homologue hermitien, l'Ensemble Unitaire Gaussien. Le second modèle concerne les matrices de permutation, où la permutation sous-jacente suit la distribution d'Ewens généralisée pour laquelle la mesure d'une permutation dépend uniquement de sa structure en cycles. Pour ces deux modèles, nous établissons la convergence en loi du polynôme caractéristique vers une fonction analytique aléatoire lorsque la dimension des matrices tend vers l'infini. Cette convergence a lieu en dehors du support des valeurs propres et est complémentaire de la convergence des distributions spectrales.

La seconde partie concerne un problème en dimension fixée. Nous considérons des produits de matrices unitaires uniformément distribuées sur des orbites de conjugaison. Nous déterminons la densité de probabilité des valeurs propres de ce produit. Cette densité est liée au volume de l'espace des modules des connexions plates sur une sphère à trois trous. Notre formule fournit une expression positive pour la densité et pour ce volume sous la forme d'une somme de volumes de polytopes explicites. Ces polytopes émergent d'objets combinatoires appelés puzzles, permettant de calculer les coefficients d'intersection pour la cohomologie des variétés de drapeaux à deux sous-espaces. Nous explorons également certaines propriétés de ces puzzles.

Mots-clés: Spectre de matrices aléatoires, fonctions analytiques aléatoires, cohomologie quantique, pavages du réseau triangulaire.

Abstract. This thesis explores certain aspects of exact solvability in random matrix theory. It is structured into two main parts.

The first part examines a high dimensional problem on the asymptotic behavior of characteristic polynomials of random matrices. We focus on two integrable models. The first is the Elliptic Ginibre Ensemble, a Gaussian interpolation between the Ginibre Ensemble and their Hermitian counterpart, the Gaussian Unitary Ensemble. The second model involves permutation matrices, where the underlying permutation follows the generalized Ewens distribution, for which the measure of a permutation only depends on its cycle structure. For both models, we establish the convergence in law of the characteristic polynomial, as the matrix dimension tends to infinity, towards a random analytic function. This convergence occurs outside the eigenvalue support and is complementary to the convergence of spectral distributions.

The second part is a fixed dimensional problem. We consider a product of unitary matrices that are uniformly distributed on fixed conjugacy orbits. We derive the probability density for the eigenvalues of this product. This probabilty density is related to the volume of moduli space of flat connections on the three-holed sphere. Our formula provides an positive expression for both the density and this volume as a sum of volumes of explicit polytopes. These polytopes arise from combinatorial objects called puzzles, which compute intersection coefficients for the cohomology of two-step flag varieties. We further investigate some properties of these puzzles.

Keywords: Spectrum of random matrices, random analytic functions, quantum cohomology, tilings of the triangular lattice.

Preprints

- Q. François and D. García-Zelada.
  Asymptotic analysis of the characteristic polynomial for the Elliptic Ginibre Ensemble.
  ArXiv 2306.16720, included as Chapter 4.

- Q. François.
  Characteristic polynomial of generalized Ewens random permutations.
  ArXiv 2504.01484, included as Chapter 5.

- Q. François and P. Tarrago.
  Positive formula for the product of conjugacy classes on the unitary group.
  ArXiv 2405.06723, included as Chapter 6.

- Q. François.
  Enumeration of crossings in two-step puzzles.
  ArXiv 2411.08412, included as Chapter 7.

Contents

Part I

Introduction

Chapter 1

Random Matrix Theory

This thesis lies in the theory of random matrices, a field that emerged from large data analysis and statistical models of heavy atoms. The origins of random matrix theory go back to the work of Wishart [Wis28] in statistical science and later to Wigner [Wig55], who introduced random matrices in the context of quantum mechanics for heavy atoms. Wigner was the first to study large-dimensional matrices, specifically in the asymptotic regime where the matrix dimension tends to infinity. Mehta had a significant impact on the field in the 1960s leading to a theory on the spectrum with the first edition of the book [Meh04]. The field continued to expand with results on the spectrum of covariance matrices by Marchenko and Pastur [MP68] in 1967. Over time, techniques from complex analysis, combinatorics and potential theory enriched random matrix theory. A major breakthrough came with the introduction of free probability by Voiculescu [Voi86] in the 1980s, leading to a better understanding of the asymptotic behavior of large random matrices. Since the 1990s, the field has witnessed advancements in the study of extreme eigenvalues, outliers and central limit theorems for linear statistics. Universality questions on the behavior of large random matrices came forward, some of which are still open.

The goal of this chapter is to give an overview of random matrix theory. This chapter is divided into four sections. Section 1.1 presents general definitions together with some observables that are relevant for the study of random matrices. It also introduces key examples of random matrices which will be ubiquitous in the rest of this thesis. Section 1.2 is devoted to Determinantal Point Processes which is a class of point processes having additional structure for its correlation functions. Determinantal point processes arise in many models related to random matrix theory to describe the joint behavior of eigenvalues. Section 1.3 gives the two main convergence results on eigenvalues for Hermitian and i.i.d. matrices that are respectively the semicircular law and the circular law. Section 1.4 goes further and presents fine fluctuation results on extreme eigenvalues and on the spectral radius. Main references for this chapter are the books [Meh04; AGZ10; BS06; MS17] and the lecture notes [Spe20].

1.1 Random Matrices

This section presents a general background on random matrices with examples of models at play in this thesis. We mainly follow [BS06].

1.1.1 Empirical Spectral Distribution of Random Matrices

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $n \geqslant 1$, we denote by $\mathcal{M}_n(\mathbb{C})$ the space of $n \times n$ matrices having complex entries.

Definition 1.1.1 (Random matrix)**.** A *random matrix* of size $n \geqslant 1$ is a random variable $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ with values in $\mathcal{M}_n(\mathbb{C})$.

A natural measure associated to a (non-necessarily random) matrix is the uniform measure on its eigenvalues. We call this measure the empirical eigenvalue distribution, or the spectral distribution, of the matrix.

Definition 1.1.2 (Empirical eigenvalue distribution)**.** Let $A \in \mathcal{M}_n(\mathbb{C})$ be a matrix of size $n \geqslant 1$. Denote by $\lambda_1(A), \ldots, \lambda_n(A)$ its eigenvalues with possible multiplicities. The *empirical eigenvalue distribution* of A is the probability measure $\mu_n(A)$ on $\mathbb{C}$ given by

$$\mu_n(A) \coloneqq \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k(A)} \ . \tag{1.1.1}$$

The empirical eigenvalue distribution is supported on at most n different atoms located on eigenvalues of A, each having weight proportionnal to the multiplicity of the associated eigenvalue.

Let us define $\mathcal{P}(\mathbb{C})$ to be the space of probability measures on $\mathbb{C}$. In the case where the matrix A_n is a random matrix of size n, the measure $\mu_n(A_n)$ is a random variable in $\mathcal{P}(\mathbb{C})$. Notice that the space $\mathcal{P}(\mathbb{C})$ does not depend on n, the size of the matrix, so that one can consider empirical eigenvalue distributions of matrices having different sizes as random variables in the same space.

1.1.2 Archetypes of random matrices

The definition of a random matrix is very broad as the entries can have arbitrary correlations and distributions. We will see that the behavior of random matrices vary depending on the structural dependence of its entries. Here, we give an overview of classical archetypes of random matrices as motivating examples.

Girko matrices

The simplest example of a random matrix is the one where the entries are independent and identically distributed random variables. Such matrices are called Girko matrices after the work of Girko [Gir18; Gir84].

Definition 1.1.3 (Girko matrix)**.** Let $A = (a_{ij})_{i,j \geqslant 1}$ be an array of i.i.d. random variables. We call the matrix $A_n = (a_{ij})_{1 \leqslant i,j \leqslant n}$ a *Girko matrix* of size n.

Definition 1.1.3 fixes the dependence relations in the random matrix but is still very general as the common law of the entries is not prescribed. The first appearance of matrices with i.i.d. entries traces back to the work of Ginibre [Gin65] who considered the particular case where entries are distributed according to the complex normal distribution. Recall that the normal or Gaussian distribution with parameters m, σ is defined as the probability distribution having density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-m)^2}{2\sigma^2} \right)$$

with respect to the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}$. We denote it by $\mathcal{N}(m, \sigma^2)$. We extend this definition to $\mathbb{C}$ by taking independent real and imaginary parts.

Definition 1.1.4 (Complex normal distribution)**.** Let Y and Z be two independent real random variables with distribution $\mathcal{N}(0, \frac{1}{2})$. The *complex normal distribution* is the law of the random variable $X = Y + iZ$. We denote it by $\mathcal{N}_{\mathbb{C}}(0, 1)$. We say that X is a standard complex Gaussian.

Another way to describe the complex normal distribution is by specifying the density $\frac{1}{\pi}\mathrm{e}^{-|z|^2}$ with respect to the Lebesgue measure $\mathrm{d}z$ on $\mathbb{C}$. One can therefore consider Girko matrices where entries follow the complex normal distribution. This specific case is known as the Ginibre Ensemble.

Definition 1.1.5 (Ginibre Ensemble)**.** The *Ginibre Ensemble* of size $n \geqslant 1$ is the law of a Girko matrix $A_n = (a_{ij})_{1\leqslant i,j\leqslant n}$ where entries a_{ij} are standard complex Gaussians. Alternatively, it is the probability distribution on $\mathcal{M}_n(\mathbb{C})$ given by

$$\mathrm{d}\mathbb{P}_n[A] \coloneqq \frac{1}{\pi^{n^2}} \exp\left(-\operatorname{Tr}\left[AA^*\right]\right) \mathrm{d}A\ , \tag{1.1.2}$$

where $\mathrm{d}A$ is the Lebesgue measure on $\mathcal{M}_n(\mathbb{C})$ and where A^* is the adjoint of A.

The Ginibre Ensemble has a rich structure. It is the first *integrable* model presented here, in the sense that working with the complex normal distribution allows for many explicit computations. The first example of such explicit computations is the probability distribution of the eigenvalues of a Ginibre matrix, obtained by Ginibre [Gin65].

Proposition 1.1.6 (Ginibre eigenvalue density, [Gin65])**.** *Let A_n be a Ginibre matrix of size $n \geqslant 1$. Then, its eigenvalues $(\lambda_1, \ldots, \lambda_n)$ have joint distribution on $\mathbb{C}^n$*

$$\frac{1}{Z_n} \prod_{i<j} |\lambda_i - \lambda_j|^2 \mathrm{e}^{-\sum_{i=1}^n |\lambda_i|^2} \mathrm{d}\lambda\ , \tag{1.1.3}$$

where $\mathrm{d}\lambda$ is the Lebesgue measure on $\mathbb{C}^n$ and where Z_n is a normalization constant.

Wigner matrices

Random matrix theory played a fundamental role in the development of quantum mechanics between 1940 and 1950. In the latter context, the motivation was to model heavy nuclei by a discretized Hamiltonian given by a $n \times n$ matrix. The discrete Hamiltonian satisfies some symmetry assumptions which give a Hermitian condition on the discretized matrix. It is therefore natural to consider random matrices satisfying a Hermitian condition. Such matrices are called Wigner matrices after the work of Wigner [Wig55]. For $n \geqslant 1$, we denote by $\mathcal{H}_n$ the space of Hermitian matrices of size n.

Definition 1.1.7 (Wigner matrix)**.** Let $A = (a_{ij})_{1\leqslant i\leqslant j}$ be an array of independent random variables and let $n \geqslant 1$. Assume that $(a_{ij})_{i<j}$ are identically distributed. We call the matrix $A_n = (a_{ij})_{1\leqslant i,j\leqslant n}$, where for $i \geqslant j$, $a_{ij} = \overline{a_{ji}}$, a *Wigner matrix* of size n.

As in Definition 1.1.3, the definition of a Wigner matrix does not specify any distribution on the upper diagonal entries of the matrix. The most celebrated example is the case where the entries strictly above the diagonal are standard complex Gaussian and the diagonal entries are standard (necessarily real) Gaussian. This particular case is called the Gaussian Unitary Ensemble (GUE).

Definition 1.1.8 (Gaussian Unitary Ensemble)**.** The *Gaussian Unitary Ensemble* of size $n \geqslant 1$ is the law of the Wigner matrix where entries a_{ij} for $j > i$ are distributed according to $\mathcal{N}_{\mathbb{C}}(0,1)$ and where a_{ii} are independent, real standard Gaussians. Alternatively, it is the probability distribution on $\mathcal{H}_n$ given by

$$\mathrm{d}\mathbb{P}[A] := \frac{1}{Z_n} \exp\left(-\frac{1}{2}\operatorname{Tr}\left[A^2\right]\right) \mathrm{d}A \ , \tag{1.1.4}$$

where $\mathrm{d}A$ is the Lebesgue measure on $\mathcal{H}_n$.

The name of the ensemble comes from the fact that the law of a GUE matrix is invariant by unitary conjugation. The Gaussian Unitary Ensemble can be seen as the Hermitian analog of the Ginibre Ensemble. Indeed, if A is a Ginibre matrix, then $X = \frac{A+A^*}{\sqrt{2}}$ is a GUE matrix. Conversely, given two independent GUE matrices X and Y, then $A = \frac{1}{\sqrt{2}}X + i\frac{1}{\sqrt{2}}Y$ is a Ginibre matrix.

As in the case of the Ginibre Ensemble, the joint law of the eigenvalues of a GUE matrix is known. Note that the Hermitian condition implies that the eigenvalues lie on the real line.

Proposition 1.1.9 (GUE eigenvalue distribution [AGZ10])**.** *Let A_n be a GUE matrix of size $n \geqslant 1$. Then, its eigenvalues $(\lambda_1, \dots, \lambda_n)$ have joint distribution*

$$\frac{1}{Z_n} \prod_{i<j} |\lambda_i - \lambda_j|^2 \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^n \lambda_i^2} \mathrm{d}\lambda \ , \tag{1.1.5}$$

where $\mathrm{d}\lambda$ is the Lebesgue measure on $\mathbb{R}^n$ and where Z_n is a normalization constant.

Coulomb gases

The two examples of the Ginibre Ensemble and the Gaussian Unitary Ensemble are quite similar as the density of the space of matrices in the two cases has the general form

$$\frac{1}{Z_n} \exp(-F(A))\mathrm{d}A \tag{1.1.6}$$

for a function F given by $F(A) = \operatorname{Tr}[AA^*]$ for Ginibre and $F(A) = \frac{1}{2}\operatorname{Tr}[A^2]$ for the GUE. The induced densities for eigenvalues share the same type of expression, where the function $x \mapsto \frac{|x|^2}{2}$ plays a central role. This motivates for a general definition of probability densities having same structure with different functions as the one above. This is known as Coulomb gas and we refer to [Cha21] for an overview on the subject. More on relation between random matrices and Coulomb gas can be found in [AGZ10; For10]

Fix $d \geqslant 1$ and let us call a measurable function $V : \mathbb{R}^d \to (-\infty, \infty]$ the potential. Define the function $g : \mathbb{R}^d \to (-\infty, \infty]$ with $g(0) = \infty$ and

$$g(x) := \begin{cases} \log(\frac{1}{|x|}) \text{ for } d = 2 \\ \frac{1}{(d-2)|x|^{d-2}} \text{ otherwise.} \end{cases} \tag{1.1.7}$$

The function g is called the Coulomb kernel. It arises from potential theory. For $\beta > 0$, $n \geqslant 1$ and $q > 0$ called the charge, the total energy $H_n : (\mathbb{R}^d)^n \to (-\infty, \infty]$ is defined as

$$H_n(x_1, \dots, x_n) := \beta q \sum_{i=1}^n V(x_i) + \beta q^2 \sum_{i<j} g(x_i - x_j) \ .$$

In electrostatics, it represents the energy of a system of n particles with charge q interacting via Coulomb interactions under an external potential V. We are now ready to define the Coulomb gas which the corresponding probability measure.

Definition 1.1.10 (Coulomb gas)**.** Let β, q be positive real numbers and $n \geqslant 1$. For $V : \mathbb{R}^d \to (-\infty, \infty]$ measurable such that

$$Z_n = \int_{(\mathbb{R}^d)^n} \mathrm{e}^{-\beta H_n(x)} \mathrm{d}x \in (0, \infty) \ ,$$

the probability measure defined by

$$\mathrm{d}\mathbb{P}_n := \frac{1}{Z_n} \mathrm{e}^{-\beta H_n(x)} \mathrm{d}x \tag{1.1.8}$$

is called the *Coulomb gas* with charge q at inverse temperature β.

The eigenvalue distribution of the Ginibre Ensemble (1.1.3) can be viewed as a two dimensional Coulomb gas with potential $V(x) = \frac{|x|^2}{2}$, unit charge at inverse temperature $\beta = 2$. The Gaussian Unitary Ensemble corresponds to a two dimensional Coulomb gaz where $V(x) = \frac{|x|^2}{4}$ for $x \in \mathbb{R} \times \{0\}$ and $+\infty$ otherwise, with unit charge and inverse temperature $\beta = 2$.

Notice that the previous densities (1.1.3) and (1.1.5) are related to the Vandermonde determinant $\prod_{i<j}(\lambda_i - \lambda_j)$ which is due to the Jacobian of the diagonalization map that appears when computing the density of eigenvalues from the density on the space of matrices, see Section 2.5 of [AGZ10]. Important examples of Coulomb gases as realizations of eigenvalues of random matrices are given by Hermite β-Ensembles that generalises the GUE density (1.1.5).

Definition 1.1.11 (Hermite β-Ensemble)**.** Fix $\beta > 0$ and $n \geqslant 1$. The Hermite β-Ensemble is the probability distribution on $(\mathbb{R})^n$ given by

$$\mathrm{d}\mathbb{P}_{\beta,n} := \frac{1}{Z_{\beta,n}} \prod_{i<j} |x_i - x_j|^\beta \mathrm{e}^{-\frac{\beta}{4}\sum_{i=1}^n x_i^2} \mathrm{d}x \ . \tag{1.1.9}$$

The Hermite β-Ensemble can be seen as a two-dimensional Coulomb gas with potential $V(x) = \frac{x^2}{2}$ for $x \in \mathbb{R} \times \{0\}$ and $+\infty$ otherwise, with unit charge, at inverse temperature β. The case $\beta = 2$ corresponds to the GUE eigenvalue density. What is remarkable is that given $\beta > 0$, one can find a random matrix model such that its eigenvalue are distributed according to (1.1.9). This result is due to Dumitriu and Edelman [DE02]. The matrix model consists of tridiagonal matrices. For a real parameter $r > 0$, let us denote by χ_r the probability distribution on $\mathbb{R}$ having density $f_r(x) = \frac{1}{\sqrt{2^{(r-2)}}\Gamma(r/2)} x^{r-1} \mathrm{e}^{-x^2/2} \mathbb{1}_{x>0}$.

Theorem 1.1.12 (Matrix representation for Hermite β-Ensemble, [DE02])**.** *Let $n \geqslant 1$, $\beta > 0$ and let $(a_{ii})_{1 \leqslant i \leqslant n}$ be i.i.d. standard Gaussian variables. Let $(a_{i,i+1})_{1 \leqslant i \leqslant n-1}$ be independent variables such that $a_{i,i+1} \sim \chi_{\beta(n-i)}$. Then, the tridiagonal, symmetric matrix*

$$A_{n,\beta} = \begin{pmatrix} \sqrt{2}a_{11} & a_{12} & & & \\ a_{12} & \sqrt{2}a_{22} & a_{21} & & \\ & \ddots & \ddots & a_{n-1,n} \\ & & a_{n-1,n} & \sqrt{2}a_{nn} \end{pmatrix}$$

has its ordered eigenvalues $\lambda_1 \leqslant \dots \leqslant \lambda_n$ distributed according to the measure (1.1.9).

Values $\beta \in \{1,2,4\}$ are of particular interest as they can be realised as eigenvalues of Gaussian random matrices. We have already seen that the case $\beta = 2$ corresponds to the Gaussian Unitary Ensemble. For $\beta = 1$, one can obtain the distribution (1.1.9) as the eigenvalue density of a real symmetric matrix where entries strictly above the diagonal are real Gaussians with variance 2 and where the diagonal entries are strandard real Gaussians. The same holds for $\beta = 4$ with quaternionic coefficients.

Unitary matrices

For $n \geqslant 1$, let us consider the unitary group

$$\mathrm{U}(n) = \{U \in \mathcal{M}_n(\mathbb{C}) \mid UU^* = U^*U = \mathrm{id}_{\mathbb{C}^n}\} .$$

This is a compact topological subgroup of $\mathcal{M}_n(\mathbb{C})$, meaning that in addition to the group structure, $\mathrm{U}(n)$ is a topological space which means that that matrix multiplication $\cdot$: $\mathrm{U}(n) \times \mathrm{U}(n) \to \mathrm{U}(n), (x,y) \mapsto x \cdot y$ and inversion $^{-1}$: $\mathrm{U}(n) \to \mathrm{U}(n), x \mapsto x^{-1}$ are continuous maps. The study of random unitary matrices was initiated by Dyson [Dys62] who considered integrable cases known as Circular Ensembles, see also the work of Girko [Gir85].
On topological groups, we have the notion of Haar measure which would be the analog of the uniform measure. We follow the line of chapter 5 in [Far08].

Definition 1.1.13 (Left invariant measure)**.** Let G be a locally compact group. A Radon measure $\mu \geqslant 0$ on G is said to be *left invariant* if, for every $h \in G$ and continuous function f on G with compact support,

$$\int_G f(hg)\mu(\mathrm{d}g) = \int_G f(g)\mu(\mathrm{d}g).$$

We give the main theorem of [Far08] Chapter 5 which states that there exists a unique left invariant measure on G up to a positive factor. This measure is called a left Haar measure on G.

Theorem 1.1.14 (Existence of Haar measure, [Far08])**.** *Any locally compact group admits a left invariant measure. Moreover, such a measure is unique up to a multiplicative factor.*

We will refer to *the* Haar measure on G as the left invariant measure μ on G with total mass of one, that is, $\mu(G) = 1$. We return to the unitary group $G = \mathrm{U}(n)$. One has an explicit construction of the Haar measure as follows. Let A_n be a Ginibre matrixx of size n. Apply the Gram-Schmidt orthonormalization procedure. Then, the law of the resulting unitary matrix with orthonormal columns is the Haar measure. The invariance under multiplication by permutation matrices, which are particular cases of unitary matrices, implies that the entries of a Haar matrix have the same distribution.

The eigenvalues of a unitary matrix are located on the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. For a random matrix U following the Haar distribution, it is natural to ask for the joint distribution of its eigenvalues. We parametrize the eigenvalues by their angles, so that $\lambda_i = \mathrm{e}^{i\theta_i}$ for $1 \leqslant i \leqslant n$ and $\theta_i \in [0, 2\pi)$. The joint density can be computed explicitely, see [HP00] for a proof.

Theorem 1.1.15 (Joint eigenvalue density of Haar unitary matrices, [HP00])**.** *For $n \geqslant 1$ and U_n distributed according to the Haar measure on $\mathrm{U}(n)$, the joint distribution of $(\theta_1, \dots, \theta_n)$ has density*

$$f(\theta_1, \dots, \theta_n) = \frac{1}{Z_n} \prod_{j<\ell} |\mathrm{e}^{i\theta_j} - \mathrm{e}^{i\theta_\ell}|^2. \tag{1.1.10}$$

The distribution (1.1.10) can also be seen as a two dimensional Coulomb gas, with unit charge, for $\beta = 2$ and with potential $V(x) = 0$ if $x \in \mathbb{S}^1$ and $V(x) = +\infty$ otherwise. As for the Hermite β-Ensemble, there exists an extension of (1.1.10) to any $\beta > 0$. Such distributions are called Circular β-Ensembles, see [DG04].

Definition 1.1.16 (Circular β-Ensemble)**.** Fix $\beta > 0$ and $n \geqslant 1$. The *Circular β-Ensemble* is the probability distribution on $[0, 2\pi)^n$ given by

$$\mathrm{d}\mathbb{P}_{\beta,n} := \frac{1}{Z_{\beta,n}} \prod_{j<\ell} |\mathrm{e}^{i\theta_j} - \mathrm{e}^{i\theta_\ell}|^\beta \mathrm{d}\theta \ . \tag{1.1.11}$$

For $\beta = 2$, we obtain the distribution of the eigenvalues of Haar unitary random matrices, also called Circular Unitary Ensemble. For $\beta = 1$ and $\beta = 4$, the distribution (1.1.11) corresponds to the eigenvalues of Haar distributed random matrices on the orthogonal and symplectic groups respectively.

Permutation matrices

This section specifies the random matrices considered here to be random permutation matrices which are special instances of orthogonal matrices. For $n \geqslant 1$, let us denote by S_n the group of permutations of $[n] = \{1, \dots, n\}$. For a permutation $\sigma \in S_n$, its associated permutation matrix is $A = A(\sigma) = (a_{ij})_{1 \leqslant i,j \leqslant n}$ where for $1 \leqslant i, j \leqslant n$,

$$a_{ij} := \mathbb{1}_{\sigma(j)=i} \ .$$

We call a permutation matrix any matrix which can be written as the associated matrix of some permutation. The cycle decomposition of a permutation $\sigma \in S_n$ is the vector $(C_k)_{1 \leqslant k \leqslant n}$ where $C_k = C_k(\sigma)$ is the number of cycles of length k in σ. For $k \geqslant 1$, let $B_k \in \mathcal{M}_k(\mathbb{C})$ be the $k \times k$ matrix associated to the permutation having one cycle $(1 \cdots k)$:

$$B_k = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} .$$

Any permutation matrix $A = A(\sigma)$ for some $\sigma \in S_n$ is conjugate to a block matrix with $C_k(\sigma)$ blocks B_k for each $1 \leqslant k \leqslant n$. Therefore, the eigenvalues of a permutation matrix $A(\sigma)$ are the roots of unity

$$\mathrm{e}^{\frac{2i\pi\ell}{k}}, 0 \leqslant \ell \leqslant k - 1$$

each occuring with multiplicity $C_k(\sigma)$.

A measure on the space of permutations yields a measure on the space of permutation matrices by the map $\sigma \mapsto A(\sigma)$. Let us introduce a measure on S_n called the Ewens measure which was first defined in [Ewe72].

Definition 1.1.17 (Ewens measure)**.** The *Ewens measure* with parameter $\theta > 0$ is the probability measure $\mathrm{d}\mathbb{P}_\theta$ on S_n given by

$$\mathrm{d}\mathbb{P}_\theta^{(n)}[\sigma] := \frac{\theta^{|\sigma|}}{Z_\theta^{(n)}}, \tag{1.1.12}$$

where $|\sigma| = \sum_{k=1}^n C_k(\sigma)$ is the number of cycles of σ and $Z_\theta^{(n)}$ is a normalization constant.

The Ewens measure specialises to the uniform measure for $\theta = 1$. It is an example of a central measure, that is, a measure which is constant on each conjugacy class of S_n since the cycle lengths are invariant by conjugation.

1.2 Determinantal Point Processes

Eigenvalues of a random matrix yield a random configuration of points in the complex plane. The mathematical definition of this object is called a point process. In Section 1.2.1, we introduce the general theory of point processes. As we are interested in eigenvalues of random matrices coming from specific models, the corresponding point process will have more structure and will often be *determinantal*, see Definition 1.2.4. This class of point processes is presented in Section 1.2.2 following the lines of [Sos00a; Bor11; Joh06] and the chapter 4 of [Hou+09].

1.2.1 Point processes

Let $\mathcal{X}$ be a Polish space, i.e a complete, separable metric space that we further assume to be locally compact. In our applications, we will consider subspaces of $\mathbb{R}^n$ for some $n \geqslant 1$. Denote by $\mathcal{B}$ its Borel sigma-algebra. An atomic measure $\mu : \mathcal{B} \to \mathbb{N} \cup \{+\infty\}$ is locally finite if for every compact subset $K \subset \mathcal{X}$, $\mu(K) < +\infty$. Such a measure can be written as $\mu = \sum_{i \in I} \delta_{x_i}$ for a countable family $(x_i)_{i \in I}$ of points in $\mathcal{X}$. Let $\mathcal{M}_a(\mathcal{X})$ be the space of atomic measures on $\mathcal{X}$. We endow $\mathcal{M}_a(\mathcal{X})$ with the smallest sigma-algebra that makes the counting applications $B \mapsto \mu(B)$ measurable for all $B \in \mathcal{B}$.

Definition 1.2.1 (Point process)**.** A *point process* on $\mathcal{X}$ is a random variable with values in $\mathcal{M}_a(\mathcal{X})$.

For a point process X and a Borel subset B, the (possibly infinite) random variable $X(B)$ counts the number of points of X which lie in B. For given $k \geqslant 1$, $B_1, \ldots, B_k \in \mathcal{B}$ and open intervals $I_1, \ldots, I_k \subset [0, \infty)$, the set of measures $\mu \in \mathcal{M}_a(\mathcal{X})$ such that for every $1 \leqslant j \leqslant k$, $\mu(B_j) \in I_j$ is called a cylinder. The sigma-algebra on $\mathcal{M}_a(\mathcal{X})$ is generated by those cylinders. To specify the law of a point process, it therefore suffices to give the joint distribution of $(X(B_j))_{1 \leqslant j \leqslant k}$ for Borel subsets $B_1, \ldots, B_k$.

A fundamental example is given by *Poisson point processes.* For μ a locally finite Borel measure on $\mathcal{X}$, the Poisson process with intensity μ is the point process defined by the following property : for every $k \geqslant 1$ and pairwise disjoint Borel subsets $B_1, \ldots, B_k$, the random variables $X(B_1), \ldots, X(B_k)$ are independent and $X(B_i)$ has Poisson distribution of parameter $\mu(B_i)$.

We will mostly work with simple point processes, that are, point processes X for which for every $x \in \mathcal{X}$, $X(\{x\}) \in \{0, 1\}$ almost surely. As random variables $(X(B))_{B \in \mathcal{B}}$ are of main importance, we introduce the correlation functions as in [Hou+09], also called the joint intensities of a point process X.

Definition 1.2.2 (Correlation functions)**.** Let X be a simple point process on $\mathcal{X}$ and let μ be a Radon measure on $\mathcal{X}$. We say that X has *correlation functions* $(\rho_k)_{k \geqslant 1}$ where $\rho_k : \mathcal{X} \to [0, \infty)$ with respect to the measure μ if for every $k \geqslant 1$ and for every pairwise disjoint subsets $B_1, \ldots, B_k \subset \mathcal{B}$,

$$\mathbb{E}\left[\prod_{i=1}^{k} X(B_i)\right] = \int_{\prod_i B_i} \rho_k(x_1, \ldots, x_k)\mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_k). \tag{1.2.1}$$

with $\rho_k(x_1, \dots, x_k) = 0$ whenever $x_i = x_j$ for some $1 \leqslant i \neq j \leqslant k$.

Example 1.2.3. An important example of a point process which admits correlation function is given by realisation of n dimensional distributions. Let $(X_1, \dots, X_n)$ be a random vector having joint symmetric density $p(x_1, \dots, x_n)$ with respect to the Lebesgue measure on $\mathbb{R}^n$. Then, the point process $X = \sum_{k=1}^n \delta_{X_k}$ on $\mathbb{R}$ has correlation functions given by

$$\rho_k(x_1, \dots, x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p(x_1, \dots, x_n) \mathrm{d}x_{k+1} \dots \mathrm{d}x_n. \tag{1.2.2}$$

Another viewpoint on correlation functions is to consider moments of variables

$$X^{\underline{n}}(B_1 \times \dots \times B_n) \coloneqq X(B_1)(X(B_2) - 1) \dots (X(b_n) - n + 1)$$

for Borel subsets $(B_i)_{1 \leqslant i \leqslant n} \in \mathcal{B}$. Recall that $\mathcal{X}$ is endowed with a given reference measure μ. Define the measure $\mu^{\underline{n}} : (B_1 \times \dots \times B_n) \mapsto \mathbb{E}[X^{\underline{n}}(B_1 \times \dots \times B_n)]$. In the case where the measure $\mu^{\underline{n}}$ is absolutely continuous with respect to the measure μ, the k-th correlation function is precisely the Radon-Nikodym derivative

$$\rho_k(x_1, \dots, x_k) = \frac{\mathrm{d}\mu^{\underline{k}}}{\mathrm{d}\mu}(x_1, \dots, x_k).$$

Correlation functions allow for computations on moments of variables $X(B)$ for $B \in \mathcal{B}$. For instance,

$$\begin{aligned} \mathbb{E}[X(B)] &= \int_B \rho_1(x) \mu(\mathrm{d}x) \\ Var[X(B)] &= \int_{B^2} \rho_2(x, y) \mu(\mathrm{d}x) \mu(\mathrm{d}y) + \int_B \rho_1(x) \mu(\mathrm{d}x) - \left(\int_B \rho_1(x) \mu(\mathrm{d}x) \right)^2. \end{aligned}$$

1.2.2 Determinantal case

The point processes that we consider here are those whose correlation functions have a determinantal form $\rho_k(x_1, \dots, x_k) = \det(K(x_i, x_j))_{1 \leqslant i,j \leqslant k}$ for a given measurable function $K : \mathcal{X}^2 \to \mathbb{C}$ called the kernel. Following [Hou+09], we make the following assumptions on the kernel K.

(i) We assume that K is locally square integrable, meaning that for any compact $C \subset \mathcal{X}$,

$$\int_{C^2} |K(x, y)|^2 \mu(\mathrm{d}x) \mu(\mathrm{d}y) < \infty.$$

(ii) K is Hermitian : $K(x, y) - \overline{K(y, x)}$ $\mu \otimes \mu$ almost every (x, y).

(iii) K is positive : $\det(K(x_i, x_j))_{1 \leqslant i,j \leqslant k} \geqslant 0$, $\mu^{\otimes k}$ almost every $(x_1, \dots, x_k)$.

(iv) The associated integral operator $\mathcal{K} : \mathrm{L}^2(\mathcal{X}, \mu) \to \mathrm{L}^2(\mathcal{X}, \mu)$ defined by

$$\mathcal{K} f : x \mapsto \int_{\mathcal{X}} K(x, y) f(y) \mu(\mathrm{d}y)$$

is locally of trace class meaning that for every compact set $C \subset \mathcal{X}$, the eigenvalues $(\lambda_j^C)_j$ of the compact restricted operator K_C on $\mathrm{L}^2(C, \mu)$ satisfy $\sum_j |\lambda_j^C| < \infty$.

Definition 1.2.4 (Determinantal Point Process)**.** Let K be a kernel satisfying the assumptions above. We say that X is a *determinantal point process* with kernel K with respect to the measure μ if X has correlation functions $(\rho_k)_{k\geqslant 1}$ such that for every $k\geqslant 1$,

$$\rho_k(x_1,\ldots,x_k)=\det(K(x_i,x_j))_{1\leqslant i,j\leqslant k}\ .$$

For a given kernel, Definition 1.2.4 does not provide any existence result of a determinantal point process with this kernel. We give a necessary and sufficient condition due to Macchi [Mac75] and Soshnikov [Sos00a], which is Theorem 4.5.5 of [Hou+09].

Theorem 1.2.5 (Existence of DPP associated to a kernel, [Hou+09])**.** *Let K be a kernel such that the associated integral operator $\mathcal{K}$ is Hermitian and locally of trace class. Let*

$$\mathrm{Spec}(\mathcal{K}):=\{z\in\mathbb{C}\mid z\cdot\mathrm{id}-\mathcal{K}\ \textit{is not invertible}\}$$

be the spectrum of $\mathcal{K}$. Then, there exists a determinantal point process with kernel K if and only if

$$Spec(\mathcal{K})\subset[0,1].$$

Before introducing examples related to random matrices, we present an important class of density functions p_n such that if $(X_1,\ldots,X_n)$ is sampled from p_n, the random point process $\sum_{i=1}^n\delta_{X_i}$ is determinantal. Let μ be a Radon measure on $\mathcal{X}$ and let $(\varphi_i)_{1\leqslant i\leqslant n}$ be orthonormal functions in $\mathrm{L}^2(\mathcal{X},\mu)$, that is, for $1\leqslant i,j\leqslant n$,

$$\int_{\mathcal{X}}\varphi_i(x)\varphi_j(x)\mu(\mathrm{d}x)=\delta_{ij}.$$

Let $p_n(x_1,\ldots,x_n)$ be the probability density given by

$$p_n(x_1,\ldots,x_n)=\frac{1}{Z_n}|\det(\varphi_i(x_j))_{1\leqslant i,j\leqslant n}|^2\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_n). \tag{1.2.3}$$

with

$$\begin{aligned}Z_n&=\int_{\mathcal{X}^n}\det(\varphi_i(x_j))_{i,j}\det(\overline{\varphi_i}(x_j))_{i,j}\mu(\mathrm{d}x_1)\ldots\mu(\mathrm{d}x_n)\\&=\sum_{\sigma,\tau\in S_n}\epsilon_\sigma\epsilon_\tau\int_{\mathcal{X}}\prod_{i=1}^n\varphi_{\sigma(i)}(x_i)\overline{\varphi_{\tau(i)}}(x_i)\mu(\mathrm{d}x_i)\\&=n!\end{aligned}$$

Proposition 1.2.6 (Symmetric density gives determinantal point process, [Hou+09])**.** *Let $p_n(x_1,\ldots,x_n)$ be a density as in (1.2.3) associated to orthonormal functions $(\varphi_i)_{1\leqslant i\leqslant n}$ and let $(X_1,\ldots,X_n)$ be a random vector with density p_n. The point process $\sum_{i=1}^n\delta_{X_i}$ is a determinantal point process with respect to the measure μ with kernel*

$$K_n(x,y)=\sum_{i=1}^n\varphi_i(x)\overline{\varphi_i}(y).$$

The proof of Proposition 1.2.6 can be found in Section 4.5 [Hou+09].

1.2.3 Examples from Random Matrix Theory

This section bridges the results from Section 1.2.2 with the examples discussed in Section 1.1.2. We now interpret the eigenvalues of random matrices as a point process via the random atomic measure

$$\sum_{i=1}^{n} \delta_{\lambda_i}.$$

We refer to this process as the eigenvalue point process.

Our focus will be on two fundamental examples that turn out to be determinantal: the eigenvalue densities of the Ginibre Ensemble (1.1.3) and the Gaussian Unitary Ensemble (1.1.5). Both of these densites involve the Vandermonde determinant

$$\Delta(x) = \Delta(x_1, \ldots, x_n) \coloneqq \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j),$$

which can be written as

$$\Delta(x) = \det(\varphi_i(x_j))_{1 \leqslant i,j \leqslant n}$$

where $(\varphi_i)_{1\leqslant i\leqslant n}$ is any family of polynomials satisfying the condition that φ_i has degree $i-1$ with leading coefficient 1. In order to recover Proposition 1.2.6, we will consider families of orthonormal polynomials associated with appropriate measures.

Gaussian Unitary Ensemble

Definition 1.2.7 (Hermite polynomial)**.** The *Hermite polynomials* $(H_n)_{n\geqslant 0}$ are the unique polynomials satisfying

- H_n has degree n and leading coefficient 1,
- They are orthonormal with respect to the Gaussian distribution,

$$\int_{\mathbb{R}} H_n(x) H_m(x) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}} \,\mathrm{d}x = n! \delta_{nm}.$$

Consider the Gaussian measure on $\mathbb{R}$ defined by $\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x$ where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}$. The Hermite polynomials are the orthogonal polynomials with respect to this measure having leading coefficient one. The density (1.1.5) of eigenvalues of a GUE matrix of size n can be written in the form (1.2.3) with orthonormal functions $(\varphi_i)_{1\leqslant i\leqslant n}$ given by

$$\varphi_i(x) = H_{i-1}(x) \frac{1}{\sqrt{(i-1)!}(2\pi)^{1/4}} \mathrm{e}^{-\frac{x^2}{4}}.$$

Therefore, from Proposition 1.2.6, the eigenvalue point process is determinantal on $(\mathbb{R}, \mathrm{d}x)$ with kernel

$$K_n^{GUE}(x,y) = \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{n} \frac{H_{i-1}(x) H_{i-1}(y)}{(i-1)!} e^{-(x^2+y^2)/4}. \tag{1.2.4}$$

Ginibre Ensemble

To derive the determinantal aspect of the eigenvalues of a Ginibre matrix which are distributed according to the density (1.1.3), we shall find the analog of the Hermite polynomials for the complex Gaussian measure $\mu(\mathrm{d}z) = \frac{1}{\pi}\mathrm{e}^{-|z|^2}\mathrm{d}z$ on $\mathbb{C}$. This is played by the monomials $(z^n)_{n\geqslant 0}$ as for $n, m \geqslant 0$,

$$\int_{\mathbb{C}} z^n \overline{z}^m \frac{1}{\pi}\mathrm{e}^{-|z|^2}\mathrm{d}z = n!\delta_{nm}.$$

Therefore, the corresponding orthonormal functions $(\varphi_i)_{1\leqslant i\leqslant n}$ in $\mathrm{L}^2(\mathbb{C},\mu)$ are

$$\varphi_i(z) = \frac{z^{i-1}}{\sqrt{\pi(i-1)!}}\mathrm{e}^{-\frac{|z|^2}{2}}$$

so that the Ginibre eigenvalue point process is determinantal with kernel

$$K_n^{Gin}(z,w) = \frac{1}{\pi}\sum_{i=1}^{n}\frac{(z\overline{w})^{(i-1)}}{(i-1)!}\mathrm{e}^{-\frac{|z|^2+|w|^2}{2}}. \tag{1.2.5}$$

1.3 Convergence of empirical eigenvalue distributions

This section presents convergence results of eigenvalue distributions as the dimension of the matrix goes to infinity. Motivations for considering such limits first appeared in the work of Wigner [Wig55; Eug58] in the context of quantum mechanics around 1950. Wigner proved that under appropriate normalization, the empirical eigenvalue distribution of GUE matrices converges on average to a limit law known as the semicircular distribution, see Theorem 1.3.10 below. This result was later extended by Arnold [Arn71; Arn67]. For Girko random matrices, the analog of the semicircular distribution is the circular law which is the uniform distribution on the unit disk in the complex plane. The convergence of the average empirical eigenvalue distribution was first established by Metha [Meh67]. for the Ginibre Ensemble while Edelman established the result for the case where entries are real Gaussians. Silverstein further strengthened these results by extending the convergence from expectation to almost sure convergence for Ginibre matrices. These early results were largely based on explicit formulas for the joint density of eigenvalues in the Gaussian case. A more general approach, aiming to extend these results beyond Gaussian ensembles to arbitrary distributions was initiated by Girko [Gir84] and further developed by Bai [Bai97]. The most general version was eventualy obtained by Tao and Vu [TV08] together with Krishnapur [TVK10].

1.3.1 Convergence of random measures

Recall that the empirical eigenvalue distribution of a matrix A_n with eigenvalues $(\lambda_i)_{1\leqslant i\leqslant n}$ is the probability measure

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i}.$$

We endow the space $\mathcal{P}(\mathbb{C})$ of probability measures on $\mathbb{C}$ with the topology of weak convergence, defined with respect to continuous and bounded functions. The weak convergence of a sequence $(\mu_n)_{n\geqslant 1}$ to a limiting measure μ is denoted by $\mu_n \implies \mu$.

Since we are dealing with random measures, we say that μ_n converges weakly to μ almost surely if

$$\mu_n \implies \mu \quad \text{almost surely,}$$

meaning that with probability one, for every continuous and bounded function f,

$$\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu.$$

For a random measure μ, we denote by $\mathbb{E}[\mu]$ the probability measure defined by

$$\mathbb{E}[\mu](B) = \mathbb{E}[\mu(B)]$$

for each borel subset B. In the case where μ is the empirical eigenvalue distribution, the measure $\mathbb{E}[\mu]$ is called the mean or average eigenvalue distribution. Note that $\mathbb{E}[\mu]$ is a deterministic measure.

1.3.2 Methods

We now describe classical methods for proving the convergence of empirical eigenvalue distributions. The first two approaches, the *moment method* and the *Stieltjes transform*, are particularly well-suited for analyzing eigenvalues of Hermitian random matrices. In Section 1.3.2, we introduce *Hermitization*, a powerful technique developed by Girko [Gir84], see also [BS06]. This method extends the scope of spectral analysis beyond Hermitian matrices.

The moment method

Let us consider the case of probability measures on $\mathbb{R}$. For $\mu \in \mathcal{P}(\mathbb{R})$, its k-th moment for $k \geqslant 1$ is given by

$$\int x^k \, \mu(\mathrm{d}x),$$

provided that $\int |x|^k \, \mu(\mathrm{d}x) < \infty$. If this condition holds for all $k \geqslant 1$, we define the sequence

$$\left(\int x^k \, \mu(\mathrm{d}x) \right)_{k \geqslant 1}$$

as the *moments* of μ. In general, for a given sequence $(m_k)_{k \geqslant 1}$ of real numbers, there may be multiple probability measures μ having this sequence as their moments. A probability measure μ is said to be *determined by its moments* if it is the unique probability measure on $\mathbb{R}$ with the moment sequence

$$\left(\int x^k \, \mu(\mathrm{d}x) \right)_{k \geqslant 1}.$$

A measure μ with compact support, meaning for some $M > 0$, $\mu([-M, M]) = 1$, is always uniquely determined by its moments. This follows from the Stone–Weierstrass theorem. For more general measures, a sufficient condition for moment determinacy is given by *Carleman's criterion*. We refer to [BS06, Lemma B.1] for this result and to Appendix B for further details on the moment method.

Lemma 1.3.1 (Carleman's criterion, [BS06])**.** *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a probability measure on $\mathbb{R}$ having moments $(m_k)_{k \geqslant 1}$. Assume that*

$$\sum_{k \geqslant 1} m_{2k}^{-\frac{1}{2k}} < \infty.$$

Then μ is uniquely determined by its moments.

The moment method relies on the following result.

Lemma 1.3.2 (Moment method, [BS06])**.** *Let $\mu \in \mathcal{P}(\mathbb{R})$ be a measure. Let $(\mu_n)_{n\geqslant 1}$ be a sequence of probability measures such that for each $n \geqslant 1$, μ_n has moments $(m_{n,k})_{k\geqslant 1}$. If*

1. *μ is uniquely determined by its moments,*
2. *$\forall k \geqslant 1 : \lim_{n\to\infty} m_{n,k} = m_k$,*

then as $n \to \infty$, $\mu_n \implies \mu$.

From Lemma 1.3.2, in order to show that almost surely, $\mu_n \implies \mu$ for a measure μ characterised by its moments, it suffices to show that moments of μ_n converge almost surely to moments of μ:

$$\forall k \geqslant 1 : \int x^k \mu_n(\mathrm{d}x) \to \int x^k \mu(\mathrm{d}x) \text{ a.s.}$$

A practical way to achieve this goal is to show that for every $k \geqslant 1$, we have

1. $\mathbb{E}[\int x^k \mu_n(\mathrm{d}x)] \to \mathbb{E}[\int x^k \mu(\mathrm{d}x)]$,
2. $Var[\int x^k \mu_n(\mathrm{d}x) - \int x^k \mu(\mathrm{d}x)] = O(n^{-2})$.

In the case where μ_n is the empirical eigenvalue distribution of a matrix A_n, its moments have an explicit expression. For $k \geqslant 1$,

$$\int x^k \mu_n(\mathrm{d}x) = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^k = \frac{1}{n}\operatorname{Tr}\left[A_n^k\right].$$

Using the moment method to show convergence of empirical eigenvalue distributions reduces to study asymptotics of traces of random matrices. For points 1. and 2. above to be computed, the entries of the random matrix should have all their moments finite, which can seem quite restrictive for our model of random matrices.
However, using a truncation technique, one is able to lower these assumptions up to necessary and sufficient conditions on first moments of the entries only. For examples of application of the truncation technique, we refer to [BS06] where it is used for various matrix models to show the convergence of empirical eigenvalue distributions.

The Cauchy transform method

This section presents a complex analysis technique used to establish the convergence of the empirical eigenvalue distribution. For more details on the Cauchy transform, we refer to Appendix B. in [BS06] and Section 3.1 in [MS17].

Definition 1.3.3 (Cauchy transform)**.** Let μ be a measure on $\mathbb{R}$. The *Cauchy transform* of μ is the complex valued function C_μ defined in the upper half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ by

$$C_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x}\mu(\mathrm{d}x),\ z \in \mathbb{C}^+.$$

The Cauchy transform is analytic on $\mathbb{C}^+$ with values in the lower half plane $\mathbb{C}^- = \{z \in \mathbb{C} : \Im(z) < 0\}$. It uniquely characterizes the measure μ by the inversion formula given in Lemma 1.3.4; see [MS17] for a proof.

Lemma 1.3.4 (Inversion formula, [MS17]). *Let μ be a probability measure on $\mathbb{R}$. For $a < b$, we have*

$$\mu((a,b)) + \mu(\{a,b\}) = -\lim_{y\to 0^+} \frac{1}{\pi}\int_a^b \Im(C_\mu(x+iy))\mathrm{d}x.$$

From this inversion formula, we deduce that if two probability measures μ and ν have the same Cauchy transform, then they must be equal:

$$C_\mu = C_\nu \implies \mu = \nu.$$

In addition to characterise the probabilty measure, the Cauchy transform is a powerful tool to derive weak convergence of a sequence of measures. For $z \in \mathbb{C}^+$, one has $\frac{1}{|z-x|} \leqslant \frac{1}{\Im(z)}$ so that $x \mapsto \frac{1}{z-x}$ is a continuous, bounded function of x from which one derives

$$\text{If } \mu_n \implies \mu \text{ then, } \forall z \in \mathbb{C}^+ : C_{\mu_n}(z) \to C_\mu(z).$$

Remarkably, the converse is also true.

Proposition 1.3.5 (Convergence of Cauchy transform implies weak convergence, [MS17]). *Let $(\mu_n)_{n\geqslant 1}$ and μ be probability measures on $\mathbb{R}$. If*

$$\forall z \in \mathbb{C}^+ : C_{\mu_n}(z) \to C_\mu(z),$$

then, $\mu_n \implies \mu$.

Thus, the problem of proving weak convergence reduces to establishing the pointwise convergence of the Cauchy transforms. If one exhibits a limiting function $C(z)$ for all $z \in \mathbb{C}^+$, it remains to check that this function is the Cauchy transform of some probability measure. A sufficient criterion for this, which can be viewed as the converse of the property

$$\lim_{y\to\infty} iyC_\mu(iy) = \mu(\mathbb{R}) = 1$$

for a probability measure μ, is given below.

Proposition 1.3.6 (Characterisation of Cauchy transforms, [MS17]). *Let $C : \mathbb{C}^+ \to \mathbb{C}^-$ be an analytic function such that $\limsup_{y\to\infty} y|C(iy)| = 1$. Then, there exists a unique probability measure μ on $\mathbb{R}$ such that $C(z) = C_\mu(z)$.*

When μ_n is the empirical eigenvalue distribution of a matrix A_n, we have the identity

$$C_{\mu_n}(z) = \int_{\mathbb{R}} \frac{1}{z-t}\mu_n(\mathrm{d}x) = -\frac{1}{n}\operatorname{Tr}[(A_n - zI_n)^{-1}].$$

For $1 \leqslant k \leqslant n$, denote by $A^{(k)}$ the matrix of size $n-1$ obtained by removing the k-th row and column of A. The matrices $(A^{(k)})_{1\leqslant k\leqslant n}$ are called the majors of A. If A and its majors are invertible,

$$\operatorname{Tr}[A^{-1}] = \sum_{k=1}^n \frac{1}{a_{kk} - \alpha_k^*(A^{(k)})^{-1}\alpha_k}$$

where α_k is the k-th row of A with entry a_{kk} removed. See Section A.1.3 in [BS06] for a proof of this result. Consequently,

$$C_{\mu_n}(z) = \sum_{k=1}^n \frac{1}{a_{kk} - \alpha_k^*(A_n^{(k)} - zI_{n-1})^{-1}\alpha_k}.$$

If one has the asymptotic $a_{kk} - \alpha_k^*(A_n^{(k)} - zI_{n-1})^{-1}\alpha_k = g(z, C_{\mu_n}(z)) + o(1)$, the limit Cauchy transform $C(z)$ can be determined by the functional equation

$$C(z) = \frac{1}{g(z, C(z))}.$$

We refer to [BS06] for applications of this technique.

Hermitization technique

The two previous methods apply to probability measures on the real line which encompass empirical eigenvalue distributions of Hermitian matrices. However, they do not extend to general matrices as their eigenvalues are generally complex. To study the empirical eigenvalue distributions of non-Hermitian matrices, Girko [Gir84] introduced a powerful technique called hermitization presented here. We refer to [BC12] and [TVK10] for more details. Let $\mathcal{P}'(\mathbb{C})$ the probability measures on $\mathbb{C}$ for which $\log|\cdot|$ is integrable at infinity.

Definition 1.3.7 (Logarithmic potential)**.** Let $\mu \in \mathcal{P}'(\mathbb{C})$. The *logarithmic potential* of μ is the function U_μ defined on $\mathbb{C}$ by

$$U_\mu(z) \coloneqq -\int_{\mathbb{C}} \log|z - x|\mu(\mathrm{d}x). \tag{1.3.1}$$

As for the Cauchy transform of probability measures on $\mathbb{R}$, the logarithmic potential characterises the measure.

Lemma 1.3.8 (Logarithmic potential characterises the measure, [BC12])**.** *Let μ and ν be two measures in $\mathcal{P}'(\mathbb{C})$ such that almost everywhere $U_\mu = U_\nu$. Then $\mu = \nu$.*

For μ_n the empirical eigenvalue distribution of a matrix A_n we have

$$U_{\mu_n}(z) = -\frac{1}{n}\log|\det(A_n - zI_n)|.$$

Using determinant properties, this can be rewritten as

$$U_{\mu_n}(z) = -\frac{1}{n}\log\det\left(\sqrt{(A_n - zI_n)(A_n - zI_n)^*}\right) = -\int \log(x)\nu_{n,z}(\mathrm{d}x),$$

where

$$\nu_{n,z} = \nu_{n,z}(A_n) \coloneqq \mu_n\left(\sqrt{(A_n - zI_n)(A_n - zI_n)^*}\right)$$

is the empirical eigenvalue distribution of the Hermitian matrix $\sqrt{(A_n - zI_n)(A_n - zI_n)^*}$. The eigenvalues of $\sqrt{AA^*}$ are called the singular values of A. Let us denote them by $s_1(A) \geqslant \cdots \geqslant s_n(A) \geqslant 0$ so that $\nu_{n,z} = \frac{1}{n}\sum_{k=1}^n \delta_{s_k(A_n - zI_n)}$. The singular values and the eigenvalues are related by

$$\prod_{k=1}^n |\lambda_k(A)| = \prod_{k=1}^n s_k(A)$$

and by the Weyl inequalities [Wey49]

$$\forall 1 \leqslant k \leqslant n : \prod_{i=1}^k |\lambda_i(A)| \leqslant \prod_{i=1}^k s_i(A).$$

We have therefore linked the logarithmic potential of a non-Hermitian matrix A with the empirical eigenvalue distribution of the Hermitian matrix $\sqrt{(A - zI)(A - zI)^*}$. The convergence of the eigenvalue distribution for non-Hermitian matrices reduces to the convergence of the eigenvalue distribution of $\sqrt{(A-zI)(A-zI)^*}$ together with some integrability condition as log is not a bounded function on $\mathbb{R}_{\geqslant 0}$. This leads to Proposition 1.3.9 below.

Proposition 1.3.9 (Hermitization, [BC12; TVK10]). *Let $(A_n)_{n\geqslant 1}$ be matrices where $A_n \in \mathcal{M}_n(\mathbb{C})$. Assume that there exists a family $(\nu_z)_{z\in\mathbb{C}}$ of probability distributions on $\mathbb{R}_{\geqslant 0}$ such that, for almost every $z \in \mathbb{C}$, almost surely,*

(i) $\nu_{n,z} \implies \nu_z$,

(ii) log *is uniformly integrable for* $(\nu_{n,z})_{n\geqslant 1}$.

Then, there exists a probability measure $\mu \in \mathcal{P}'(\mathbb{C})$ such that

(iii) almost surely, $\mu_n \implies \mu$

(iv) almost everywhere, $U_\mu(z) = -\int_0^\infty \log(x)\nu_z(\mathrm{d}x)$.

1.3.3 Wigner matrices and the semicircular law

In this section, we restrict our attention to Wigner matrices as in Definition 1.1.7. We give a general version of convergence of the empirical eigenvalue distribution for such matrices which can be found in [BS06].

Theorem 1.3.10 (Wigner's semi-circular law, [BS06]). *Let A_n be a Wigner matrix of size n such that entries above the diagonal have unit variance. Assume that all entries are centered. Then, almost surely,*

$$\mu_n\left(\frac{1}{\sqrt{n}}A_n\right) \implies \mu_{s.c}$$

where

$$\mu_{s.c}(\mathrm{d}x) := \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{1}_{|x|<2}\mathrm{d}x.$$

The limit probability distribution $\mu_{s.c}$ is called the semi-circular distribution, see Figure 1.1. Its Cauchy transform is given by $C_{\mu_{s.c}}(z) = \frac{z-\sqrt{z^4-4}}{2}$. Theorem 1.3.10 is the first example of *universality* encountered as it holds for arbitrary pairs of distributions for the diagonal and above diagonal entries as long as the former has unit variance.

Remark 1.3.11. Let us make a few comments.

1. In the case where the entries above the diagonal have a general variance $\sigma^2 > 0$, the limit distribution obtained is the scaled semicircular law with density

$$\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-x^2}\mathbb{1}_{|x|<2\sigma}\mathrm{d}x.$$

2. The assumption that all entries are identically distributed can be relaxed. Consider a Wigner matrix where the diagonal and above diagonal entries are independent but not necessarily with the same law and such that the law of each entry might depend on n. Together with the condition

$$\forall\eta > 0 \lim_{n\to\infty}\frac{1}{n^2}\sum_{j,k=1}^{n}\mathbb{E}\left[|a_{jk}^{(n)}|^2\mathbb{1}_{|a_{jk}^{(n)}|\geqslant\eta\sqrt{n}}\right],$$

the conclusion of Theorem 1.3.10 holds.

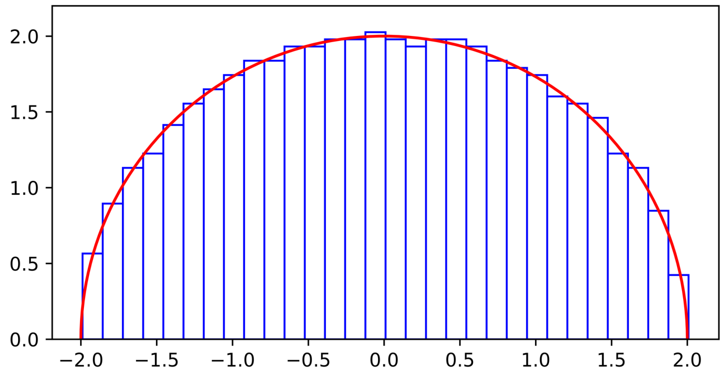

Figure 1.1: Illustration of Theorem 1.3.10. Eigenvalues of a scaled GUE matrix of size 10^3. The semicircular density is plotted in red.

3. The second order moment condition for the above diagonal entries is necessary and also sufficient in Theorem 1.3.10. In the case of heavy tail entries, the empirical eigenvalue distribution converges to other distributions depending on the parameter of the stable law, see [BC94], [BDG09] and [BCC11].

1.3.4 Girko matrices and the circular law

This section presents the analog of Wigner's theorem for Girko matrices, that is, matrices with i.i.d. entries without the Hermitian condition. The general version below can be found in [TVK10], see also [BC12] for more details on this result.

Theorem 1.3.12 (Circular law, [TVK10]). *Let $A_n = (a_{ij})_{1\leqslant i,j\leqslant n}$ where $(a_{ij})_{i,j\geqslant 1}$ are i.i.d. random variables with $\mathbb{E}[a_{ij}] = 0$ and $\mathbb{E}[|a_{ij}|^2] = 1$. Then, almost surely,*

$$\mu_n\left(\frac{1}{\sqrt{n}}A_n\right) \implies \mu_c, \tag{1.3.2}$$

where

$$\mu_c(\mathrm{d}z) := \frac{1}{\pi}\mathbb{1}_{|z|<1}\mathrm{d}z.$$

The probability distribution $\mu_c(\mathrm{d}z) = \frac{1}{\pi}\mathbb{1}_{|z|<1}\mathrm{d}z$ is called the circular law. It can be seen as the non-Hermitian analog of the semicircular distribution.
Let $\nu_{n,z} = \nu_{n,z}\left(\frac{A_n}{\sqrt{n}}\right)$. The proof of Theorem 1.3.12 relies on Proposition 1.3.9 for which one has to show the existence of probability distributions $(\nu_z)_{z\in\mathbb{C}}$ such that almost surely and almost everywhere, $\nu_{n,z} \implies \nu_z$, together with uniform integrability condition of $\log|.|$ for $(\nu_{n,z})_{n\geqslant 1}$. For more details on the first part, we refer to Section 4.5 of [BC12]. To prove uniform integrability, it is sufficient to show that for $z \in \mathbb{C}$, one can find $p > 0$ such that almost surely

$$\limsup_{n\to\infty} \int x^{-p}\nu_{n,z}(\mathrm{d}x) < \infty \text{ and } \limsup_{n\to\infty} \int x^{p}\nu_{n,z}(\mathrm{d}x) < \infty. \tag{1.3.3}$$

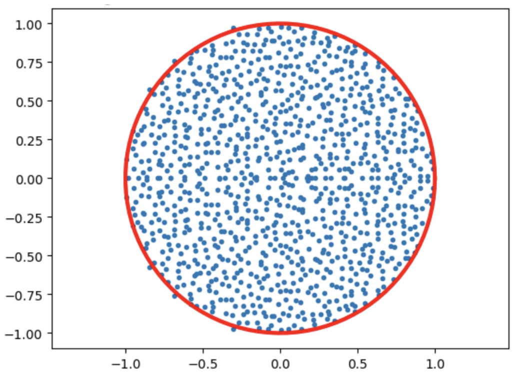

Figure 1.2: Illustration of Theorem 1.3.12. Eigenvalues of a scaled Ginibre matrix of size 500. The unit circle is plotted in red.

The second condition can be ensured for $p = 2$ using that for $1 \leqslant i \leqslant n$, $s_i\left(\frac{1}{\sqrt{n}}A_n - zI_n\right) \leqslant s_i\left(\frac{1}{\sqrt{n}}A_n\right) + |z|$ and that by the law of large numbers, almost surely as $n \to \infty$,

$$\int x^2 \nu_{n,z}\left(\frac{1}{\sqrt{n}}A_n\right) \mathrm{d}x = \frac{1}{n^2}\operatorname{Tr}[A_n A_n^*] = \frac{1}{n^2}\sum_{1 \leqslant i,j \leqslant n} |A_{ij}|^2 \to \mathbb{E}\left[|A_{11}|^2\right].$$

The first condition is the most challenging to establish as it involves a detailed analysis of the smallest singular values of non-Hermitian random matrices. This analysis was carried out in [TVK10; TV08].

1.4 Convergence of extreme eigenvalues

The previous Section 1.3 established results on the convergence of the empirical eigenvalue distribution, which can be interpreted as a global characterization of the eigenvalue spectrum. However, this does not necessarily imply that the spectral radius, or in the case of Hermitian matrices, the largest eigenvalue, converges to the edge of the limiting distribution. The work on extreme eigenvalues of sample covariance matrices was initiated by Geman [Gem80], who showed the convergence of the largest eigenvalue under moment conditions on the entries. Later, Bai, Krishnaiah, and Yin [YBK88] established that a fourth-moment condition is sufficient for this convergence. The fourth order moment was also proved to be necessary in [BSY88] as, if it is infinite, the limsup of the largest eigenvalue tends to infinity almost surely. For Wigner matrices with real entries, Bai and Yin [BY88] proved that the fourth order moment condition is both necessary and sufficient for the largest eigenvalue to converge to the boundary of the semicircular distribution. This result was later extended to all Wigner matrices with complex entries by Bai and Silverstein [BS06] as presented in Theorem 1.4.3.

1.4.1 Wigner matrices and the Tracy Widom law

This section is dedicated to main results on the convergence of the largest eigenvalue of a Wigner matrix to the edge of the semicircular distribution. We refer the reader to chapter 5 of [BS06] for more details on the results presented. In this section, A_n is a Wigner matrix of size $n \geqslant 1$.

Convergence of the largest eigenvalue

Denote by

$$\rho_n = \rho(A_n/\sqrt{n}) := \frac{1}{\sqrt{n}} \max_{1\leqslant k\leqslant n} |\lambda_k(A_n)|$$

the spectral radius of $\frac{1}{\sqrt{n}}A_n$, that is, the eigenvalue with the largest modulus. Under the assumptions of Theorem 1.3.10, we know that when entries above the diagonal having unit variance,

$$\liminf_{n\to\infty} \rho_n \geqslant 2.$$

Theorem 1.4.1 shows that under moment assumptions, the largest eigenvalue converges to the boundary value 2 of the semicircular distribution.

Theorem 1.4.1 (Extreme eigenvalues of Wigner matrices, [BS06])**.** *Let A_n be a Wigner matrix and let $\sigma^2 > 0$. The conditions*

(i) $\mathbb{E}[a_{11}^2] < \infty$,

(ii) $\mathbb{E}[a_{12}] = 0$,

(iii) $\mathbb{E}[|a_{12}|^2] = \sigma^2$,

(iv) $\mathbb{E}[|a_{12}|^4] < \infty$,

are equivalent to the almost sure convergence

(j) $\rho_n \to 2\sigma$.

By a symmetry argument, under the same assumptions of Theorem 1.4.1, one can show that the smallest eigenvalue $\frac{1}{\sqrt{n}} \min_{1\leqslant k\leqslant n} \lambda_k(A_n)$ converges almost surely to -2σ.

Condition (iv) of Theorem 1.4.1 is the main difference from the assumptions of Theorem 1.3.10 which only assumed a second order moment condition. One can derive the convergence of the largest eigenvalue under weaker assumptions than the fourth order moment for real-valued entries above the diagonal satisfying certain tail conditions. However, the convergence to 2σ will only hold weakly or equivalently, in probability, see Theorem 5.3 in [BS06].

Tracy-Widom fluctuations

The next step on the convergence of the largest eigenvalue of Wigner matrices is to know the fluctuations around the limit value 2σ of Theorem 1.4.1. This can be seen as analogous to the central limit theorem, which follows after the law of large numbers. However, unlike the Gaussian distribution, the fluctuations in this case follow the Tracy-Widom distribution introduced in Definition 1.4.2. It is named after Tracy and Widom who obtained estimates for convergence rate and fluctuations for GUE matrices [TW94]. In [TW96], they extended their result to β-Hermite ensembles for $\beta \in \{1, 2, 4\}$ covering orthogonal and symplectic ensembles. The extension to general Wigner matrices, known as *edge universality*, is due to various contributions including [Sos99; PS07; TV10] and [LY14].

As we want to study the behavior around the edge of the semicircular distribution, we expect, as discussed in [Spe20], that the number of eigenvalues in $[2-t, 2]$ for $t > 0$ behaves as

$$n \int_{2-t}^{2} \frac{1}{2\pi}\sqrt{4-x^2}\mathrm{d}x \sim \frac{2n}{3\pi} t^{3/2}$$

as $t \to 2$. The appropriate scaling suggested by this expression is $t \sim n^{-2/3}$ so that one should consider $n^{2/3}(\rho_n - 2)$ in order to see non trivial fluctuations. This scaling corresponds to the correct formulation of the Tracy-Widom fluctuations.

The Tracy-Widom distribution is defined in terms of the Airy function, $\mathrm{Ai} : \mathbb{R} \to \mathbb{R}$ which is the solution to the differential equation

$$u''(t) = tu(t)$$

having asymptotic behavior

$$\mathrm{Ai}(t) \sim \frac{1}{2\sqrt{\pi}} t^{-1/4} \mathrm{e}^{-\frac{2}{3}t^{-3/2}} \text{ as } t \to \infty.$$

Definition 1.4.2 (Tracy-Widom distribution)**.** Let $q : \mathbb{R} \to \mathbb{R}$ be the solution to the differential equation $q(t)'' = tq(t) + 2q^3(t)$ satisfying $q(t) \sim \mathrm{Ai}(t)$ as $t \to \infty$. The *Tracy-Widom distribution* is the probability distribution on $\mathbb{R}$ with cumulative distribution function

$$F_{TW}(x) := \exp\left(-\int_x^\infty (t-x)q^2(t)\mathrm{d}t\right).$$

We now present the general form of edge universality as given in [LY14]. This stronger version of fluctuation analysis provides a necessary and sufficient condition for the convergence of joint distribution function of the k largest eigenvalues, yielding the Tracy-Widom distribution for the largest eigenvalue by taking $k = 1$.

Theorem 1.4.3 (Edge universality for Wigner matrices, [LY14])**.** *Let A_n be a Wigner matrix such that entries above the diagonal have unit variance and the diagonal entries have finite variance. Let the eigenvalues of $\frac{1}{\sqrt{n}}A_n$ be ordered as $\lambda_n \leqslant \cdots \leqslant \lambda_1$. The following statements are equivalent*

(i) $\lim_{t\to\infty} t^4 \cdot \mathbb{P}[|a_{12}| > t] = 0$

(ii) For any fixed $k \geqslant 1$, the joint distribution function

$$\mathbb{P}\left[n^{2/3}(\lambda_1 - 2) \leqslant t_1, \ldots, n^{2/3}(\lambda_k - 2) \leqslant t_k\right]$$

converges to the Tracy-Widom joint distribution.

Note that the assumption (i) of Theorem 1.4.3 is strictly stronger than the fourth-order moment condition in Theorem 1.4.1 which ensures the convergence of the largest eigenvalue to 2. As shown in [LY14], if this condition does not hold, then

$$\limsup_{n\to\infty} \mathbb{P}[\rho_n \geqslant 3] > 0.$$

1.4.2 Girko matrices and the Gumbel law

This section addresses the convergence of extreme eigenvalues for i.i.d. matrices. For the rest of this section, A_n denotes a Girko matrix with centered entries.

Convergence of the spectral radius

Denote by ρ_n the spectral radius of $\frac{1}{\sqrt{n}}A_n$. Under the assumptions of Theorem 1.3.12, the circular law implies that

$$\liminf_{n\to\infty} \rho_n \geqslant 1.$$

The analogous question in this setting, relative to the convergence of the largest eigenvalue in Wigner matrices, concerns the convergence of the spectral radius ρ_n. The convergence of the spectral radius for i.i.d. matrices was first established by Mehta [Meh04] in the integrable case of the Ginibre Ensemble, see chapter 15 therein. The result was later extended by Geman [Gem86] to entries with moments bounded by polynomials and subsequently by Bai and Yin [BY86] to entries having finite fourth moment. The fourth order moment condition was later reduced by Bordenave, Chafaï and García-Zelada [BCG22] to an optimal second-order moment condition. We present this latter result as its proof techniques are closely related to methods discussed in Chapters 2, 4 and 5 of this thesis.

Theorem 1.4.4 (Convergence of the spectral radius of Girko matrices, [BCG22])**.** *Let A_n be a Girko matrix whose entries have unit variance. Then, in probability,*

$$\lim_{n\to\infty} \rho_n = 1. \tag{1.4.1}$$

Gumbel fluctuations

Similar to the Wigner matrix case, one can study the fluctuations of the spectral radius around the limit 1. The counterpart to the Tracy-Widom distribution in this context is the Gumbel distribution.

Definition 1.4.5 (Gumbel distribution)**.** The Gumbel distribution is the probability distribution on $\mathbb{R}$ with cumulative distribution function

$$F_G(x) := \mathrm{e}^{-\mathrm{e}^{-x}}.$$

Rider [Rid03] proved that the spectral radius of Ginibre matrices has Gumbel fluctuations. A first step towards the universality of Gumbel fluctuations for the spectral radius was proved for two-dimensional Coulomb gases with a radially symmetric potential in [CP14]. Note that for such distributions over random matrix ensembles, the limit empirical eigenvalue distribution is supported on a centered annulus in the complex plane which is more general than the unit circle corresponding to the quadratic potential, that is, for the Ginibre Ensemble. In a recent work, Cipolloni, Erdős and Xu [CEX23] established the edge universality of Gumbel fluctuations for i.i.d. matrices. We state their main result as Theorem 1.4.6.

Theorem 1.4.6 (Gumbel fluctuations for i.i.d. matrices, [CEX23])**.** *Let $\gamma_n = \log(n/2\pi) - 2\log\log(n)$ and let A_n be a matrix of size n with i.i.d. complex entries. Assume that the entries satisfy $\mathbb{E}[a_{11}] = 0$, $\mathbb{E}[|a_{11}|^2] < \infty$ and $\mathbb{E}[a_{11}^2] = 0$. Moreover, assume that entries have finite moments:*

$$\forall p \geqslant 0 : \mathbb{E}[|a_{11}|^p] \leqslant C_p \tag{1.4.2}$$

for some constants $(C_p)_{p\geqslant 1}$. Then, as $n \to \infty$, the scaled spectral radius

$$\sqrt{4n\gamma_n}\left(\rho_n - 1 - \sqrt{\frac{\gamma_n}{4n}}\right) \tag{1.4.3}$$

converges in distribution to a Gumbel distributed random variable.

In [CEX23], they furthermore prove that the argument of the largest eigenvalue converges in distribution to a random variable following the uniform law on the unit circle independent of the modulus. Their result also proves the convergence of moments of (1.4.3) to moments of the Gumbel distribution. The authors further argue that the moment condition (1.4.2) can be relaxed to finiteness up to some sufficiently large p_0.

1.4.3 Wigner to Girko interpolation

We conclude Section 1.4 by mentioning an interpolation due to Johansson [Joh07] which unifies the two limiting distributions for the extreme eigenvalue: the Tracy-Widom distribution from Wigner matrices and the Gumbel law from Girko matrices. This interpolation relies on a one-parameter determinantal point process for which the distribution of the last particle interpolates between the Tracy-Widom and the Gumbel distributions. For $\alpha > 0$, define the kernel $K_\alpha : \mathbb{R}^2 \to \mathbb{R}$ by

$$K_\alpha(x, y) := \int_{\mathbb{R}} \frac{e^{\alpha t}}{e^{\alpha t} + 1} \operatorname{Ai}(x + t)\operatorname{Ai}(y + t)\, \mathrm{d}t.$$

The kernel K_α is related to the distribution of the KPZ equation [ACQ11], and to multiplicative statistics of the Airy determinantal random point process [BG16]. The kernel K_α defines a trace class operator in $\mathrm{L}^2(a, \infty)$. Let X_α be a determinantal process with kernel K_α. As for any $t \in \mathbb{R}$, $\int_t^\infty K_\alpha(x, x)\mathrm{d}x < \infty$, one can compute the cumulative distribution function of the largest particle which is given by the Fredholm determinant formula

$$F_\alpha(t) = \mathbb{P}[X((t, \infty)) = 0] = \sum_{n \geqslant 0} \frac{(-1)^n}{n!} \int_{(t,\infty)^n} \det(K_\alpha(x_i, x_j))_{1 \leqslant i,j \leqslant n} \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

The main result of [Joh07] is the following pair of convergences of the cumulative distribution function for the last particle of X_α to Gumbel and Tracy-Widom ones.

Proposition 1.4.7 (Gumbel to Tracy-Widom interpolation, [Joh07])**.** *Let F_G and F_{TW} be the cumulative distribution functions of the Gumbel and Tracy-Widom laws respectively. For every $x \in \mathbb{R}$,*

$$\lim_{\alpha \to 0^+} F_\alpha\left(\frac{x}{\alpha} - \frac{3}{2\alpha}\log(4\pi\alpha)\right) = F_G(x),$$
$$\lim_{\alpha \to \infty} F_\alpha(x) = F_{TW}(x).$$

Chapter 2

Characteristic polynomials

This chapter focuses on the characteristic polynomials of random matrices. Unlike the previous chapter which examines the eigenvalue point process, this approach treats the characteristic polynomial as a random function. Under appropriate normalization, the goal is to establish its convergence to a well-defined limit analytic function.

The study of characteristic polynomials in random matrix theory serves two main purposes. The first approach starts with a given random matrix model and proves the convergence of its characteristic polynomial. This, in turn, provides insight into the spectral properties of the matrix. Beyond spectral analysis, limits of characteristic polynomials involve random functions that are of independent interest. For instance, in the case of centered Girko matrices, the limiting function is the exponential of a Gaussian planar function which has been studied independently, see [Hou+09].

A second perspective involves using characteristic polynomials to reinterpret existing problems by connecting them to well-chosen random matrices. This approach was used by Keating and Snaith [KS00], who related the characteristic polynomial of Haar unitary matrices to the Riemann Zeta function. Their work led to a conjecture on the moments of the Zeta function which was motivated by Montgomery's conjecture [Mon73] in analytic number theory and the work of Rudnick and Sarnak [RS96] on the connection between zeros of L-functions and random matrices. Characteristic polynomials also appear in various other contexts, notably in statistical physics where they are related to log-correlated gases and Gaussian fields, see [BK22] for an overview.

2.1 Random characteristic polynomials and traces

The central object in this chapter is the characteristic polynomial

$$p_n(z) \coloneqq \det\left(I_n - zA_n\right)$$

of a random matrix A_n. We aim to study its convergence as a random variable taking values in the space of holomorphic functions endowed with the topology of local uniform convergence. The coefficients $(c_k^{(n)})_{0\leqslant k\leqslant n}$ of p_n defined via

$$p_n(z) = \sum_{k=0}^{n} c_k^{(n)} z^k,$$

are known as the secular coefficients. They are related to traces of powers of A_n by

$$c_k^{(n)} = P_k\left(\mathrm{Tr}[A_n], \dots, \mathrm{Tr}\left[A_n^k\right]\right)$$

where $(P_k)_{k\geqslant 0}$ is a family of polynomials independent of n. Therefore, the study of secular coefficients and characteristic polynomials can be done via the convergence of traces $\left(\mathrm{Tr}\left[A_n^k\right]\right)_{k\geqslant 1}$. One can further highlight the relation with traces by expanding the logarithm

$$\log p_n(z) = -\sum_{k\geqslant 1} \frac{z^k}{k} \mathrm{Tr}\left[A_n^k\right] \tag{2.1.1}$$

as a formal identity. If one shows the joint convergence of traces in (2.1.1) to some family of coefficients, a natural candidate for the limiting function would be the analytic function having these coefficients. This point of view has been used in various contexts, see for instance [BCG22; Cos23; CLZ24; NPS23]. As we will see in the results below, the limiting functions for characteristic polynomials of large matrices often exhibit structured form involving the exponential of a random power series.

2.2 Unitary matrices

We begin with characteristic polynomial of unitary matrices. For $A_n \in \mathrm{U}(n)$, its eigenvalues are located on the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Recall the notation $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ for the open unit disk in the complex plane.

2.2.1 Circular Ensembles

The first result we provide concerns eigenvalues sampled from the Circular β-Ensemble distribution (1.1.11). When $\beta = 2$, the distribution corresponds to the eigenvalue distribution of Haar-distributed unitary matrices. Motivated by the logarithmic expansion (2.1.1), a natural approach to studying the convergence of the characteristic polynomial is through the convergence of traces. Convergence in law for traces were first established for Haar matrices by Diaconis and Shahshahani [DS94], and later extended to any values of $\beta > 0$ by Jiang and Matsumoto [JM15], using tools from symmetric function theory.

Theorem 2.2.1 (Convergence of traces for CβE, [JM15])**.** *Let $\ell \geqslant 1$ and let A_n be a unitary matrix whose eigenvalues follow the Circular β-Ensemble distribution* (1.1.11)*. Then, as $n \to \infty$, we have the convergence in distribution:*

$$\left(\mathrm{Tr}\left[A_n\right], \dots, \mathrm{Tr}\left[A_n^\ell\right]\right) \to \sqrt{\frac{2}{\beta}}(X_1, \sqrt{2}X_2, \dots, \sqrt{\ell}X_\ell),$$

where $(X_k)_{1\leqslant k\leqslant \ell}$ are i.i.d. standard complex Gaussians.

This result suggests a candidate for the limiting characteristic polynomial of the Circular β-Ensemble:

$$\exp\left(\sqrt{\frac{2}{\beta}}\sum_{k\geqslant 1}\frac{z^k}{\sqrt{k}}X_k\right) \tag{2.2.1}$$

where $(X_k)_{k\geqslant 1}$ are i.i.d. standard complex Gaussians. The function

$$f(z) = \sum_{k\geqslant 1}\frac{z^k}{\sqrt{k}}X_k \tag{2.2.2}$$

is a particular example of a *Gaussian analytic function* which are ubiquitous for limits of characteristic polynomials. We provide a general definition that can be found in [Hou+09].

Definition 2.2.2 (Gaussian Analytic Function, [Hou+09])**.** Let $\Lambda \subset \mathbb{C}$ be a region in the complex plane. Let f be random variable with values in $\mathcal{H}(\Lambda)$, the space of analytic functions on Λ. Then, f is said to be *Gaussian Analytic Function* if for all $n \geqslant 1$ and points $(z_1, \ldots, z_n) \in \Lambda^n$, the vector

$$(f(z_1), \ldots, f(z_n))$$

is a centered complex Gaussian vector in $\mathbb{C}^n$.

Inside the unit disk

The expression (2.2.2) is well defined on the unit disk $\mathbb{D}$. Najnudel, Paquette and Simm [NPS23] established the convergence of the characteristic polynomial to the function (2.2.1).

Theorem 2.2.3 (Convergence of CβE characteristic polynomial, [NPS23])**.** *For $\beta > 0$, one can construct a probability space such that for any $r \in (0, 1)$, one has almost surely as $n \to \infty$,*

$$\sup_{|z|<r} \left| p_n(z) - \mathrm{e}^{\sqrt{\frac{2}{\beta}} \sum_{k \geqslant 1} \frac{z^k}{\sqrt{k}} X_k} \right| \to 0.$$

where $(X_k)_{k \geqslant 1}$ are i.i.d. standard complex Gaussians.

The limiting function

$$F(z) = \mathrm{e}^{f(z)}$$

where f is a Gaussian analytic function is referred as the *holomorphic multiplicative chaos* as introduced in [NPS23]. See also [Naj+25] for further results on its Fourier coefficients. It is an example of log-correlated field, in the sense that for $z, w \in \mathbb{D}$, the covariance structure of f is given by

$$\mathbb{E}[f(z)f(w)] = 0 \text{ and } \mathbb{E}[f(z)\overline{f(w)}] = -\log(1 - z\overline{w}).$$

The convergence of Theorem 2.2.3 holds uniformly on compact subsets of the unit disk $\mathbb{D}$. This corresponds to a region which does not contain any eigenvalues, which are all located on the unit circle. In contrast, the study of the characteristic polynomial in the critical region, that is, on the unit circle, yields other types of limiting behaviors which we explore in the next paragraph.

On the unit circle

A Gaussian limit for the log-characteristic polynomial of Haar unitary matrices evaluated on the unit circle was first identified in [HKO01]. The scaling is different from the one of 2.2.3 due to the presence of eigenvalues. This difference had already been observed in the earlier work of Keating and Snaith [KS00], where the authors proved the convergence in distribution

$$\frac{\log p_n(1)}{\sqrt{\frac{1}{2}\log(n)}} \to \mathcal{N}_{\mathbb{C}}(0, 1).$$

This suggested the right scaling later used in [HKO01].

Chhaibi, Najnudel and Nikeghbali [CNN17] considered ratios of characteristic polynomial of Haar random matrices in the so-called microscopic regime, near the point $1 \in \mathbb{S}^1$ corresponding to the scaling z/n.

Theorem 2.2.4 (Convergence of ratios [CNN17])**.** *Let A_n be a Haar distributed unitary matrix and let us define*

$$\xi_n(z) := \frac{\det\left(I_n - e^{\frac{2i\pi z}{n}} A_n^*\right)}{\det\left(I_n - A_n^*\right)}.$$

Then, as $n \to \infty$, we have the convergence in law, for the topology of local uniform convergence in $\mathbb{C}$,

$$\xi_n \to \xi_\infty,$$

where

$$\xi_\infty(z) := e^{i\pi z} \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{y_k}\right)$$

and where $(y_k)_{k\in\mathbb{Z}}$ are sampled from the determinantal point process called the Sine process having kernel on $\mathbb{R}^2$

$$K(x,y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}.$$

The same function ξ_∞ was identified to be the limit for ratios of characteristic polynomials beyond the Haar unitary case in the work [Chh+19]. It is the limit for ratios of characteristic polynomial for matrices sampled from the Haar measure on the special orthogonal group $\mathrm{SO}(n)$ and the symplectic group $\mathrm{Sp}(n)$. The same work also establishes that the limit of ratios of characteristic polynomials of GUE matrices with a different scaling involves the same entire function.

2.2.2 Permutation matrices

In this section, we consider the special case of permutation matrices. Let $A_n = A(\sigma_n)$ be a permutation matrix associated to a permutation $\sigma_n \in S_n$. By decomposing the permutation σ in disjoint cycles, the characteristic polynomial of its permutation matrix can be expressed accordingly. If σ_n has cycle decomposition $(C_1^{(n)}, \dots, C_n^{(n)})$, its characteristic polynomial is given by

$$p_n(z) = \det(I_n - zA_n) = \prod_{k=1}^{n} \left(1 - z^k\right)^{C_k^{(n)}}. \tag{2.2.3}$$

As for circular ensembles, there are two main regimes for analyzing the characteristic polynomial depending on the location of the complex variable z. The first approach examines the characteristic polynomial on the unit circle, where eigenvalues are located. The second deals with z inside the unit disk $\mathbb{D}$ where it does not vanish and where it is shown to converge to a random analytic function analogous to the result in Theorem 2.2.3.

On the unit circle

In the same vein as for Haar unitary matrices, the log-characteristic polynomial of uniform permutation matrices was shown to converge to a Gaussian limit by Hambly, Keevash, O'Connell and Stark [Ham+00]. This convergence holds at points $e^{2i\pi\alpha}$ on the unit circle where the angle $\alpha \in (0,1)$ is irrational and of finite type. We refer to [Ham+00] for the precise definition and only state their main result which exhibits a $\sqrt{\log(n)}$ scaling, similar to the case of Haar random matrices.

Theorem 2.2.5 (Characteristic polynomial of uniform permutation matrices, [Ham+00])**.** *Let $A_n = A_n(\sigma)$ be a random permutation matrix where σ is uniformly distributed on S_n. Let $\alpha \in (0,1)$ be an irrational number of finite type. Then, as $n \to \infty$, the real and imaginary parts of*

$$\frac{\log p_n(\mathrm{e}^{2i\pi\alpha})}{\sqrt{\frac{\pi^2}{12}\log(n)}}$$

converge to standard real Gaussian random variables.

Zeindler [Zei13] further showed that the two limiting Gaussians in Theorem 2.2.5 are independent so that the limit of the log-characteristic polynomial is a standard complex Gaussian. Zeindler's results are more general as they remain valid for the Ewens distribution (1.1.12) with any parameter $\theta > 0$. Together with Dang [DZ14], these results were extended to describe the joint convergence of the log-characteristic polynomial for Ewens random matrices at several irrational points on the unit circle.

Theorem 2.2.6 (Convergence of log-characteristic polynomial for Ewens permutations, [DZ14])**.** *Let $k \geqslant 1$ and let $(\mathrm{e}^{2i\pi\alpha_1}, \dots, \mathrm{e}^{2i\pi\alpha_k})$ be points on $\mathbb{S}^1$ with irrational angles $(\alpha_i)_{1\leqslant i\leqslant k}$ pairwise of finite type, see [DZ14]. Let A_n be a random permutation matrix following the Ewens distribution with parameter $\theta > 0$. Then, we have the convergence in law*

$$\frac{1}{\sqrt{\frac{\pi^2}{12}\theta\log(n)}}\begin{pmatrix}\log p_n(\mathrm{e}^{2i\pi\alpha_1})\\ \vdots \\ \log p_n(\mathrm{e}^{2i\pi\alpha_k})\end{pmatrix} \underset{n\to\infty}{\to} \begin{pmatrix}Z_1\\ \vdots\\ Z_k\end{pmatrix},$$

where $Z_1, \dots, Z_k$ are independent complex Gaussians, each with independent, centered real and imaginary parts of unit variance.

A natural question is whether this result can be extended to general circular ensembles as defined in (1.1.11).

Inside the unit disk

Similar to the results of [NPS23] for the characteristic polynomial of circular ensembles, Coste, Lambert and Zhu [CLZ24] studied the characteristic polynomial of permutation matrices under the uniform distribution inside the unit disk. Since permutation matrices have no eigenvalues in this region, one can expect a convergence towards a random analytic function.

Theorem 2.2.7 (Convergence of the characteristic polynomial of uniform permutations, [CLZ24])**.** *Let $d \geqslant 1$ be a fixed integer and let A_n be a sum of d i.i.d. uniform permutation matrices. Let $(\Lambda_\ell)_{\ell\geqslant 1}$ be independent Poisson random variables with respective parameters $(\frac{d^\ell}{\ell})_{\ell\geqslant 1}$. Then,*

$$\frac{1}{\sqrt{d}}p_n\left(\frac{z}{\sqrt{d}}\right)$$

converges in law, for the topology of local uniform convergence in $\mathbb{D}$ to

$$\left(z - \frac{1}{\sqrt{d}}\right)\frac{\mathrm{e}^{-Y_d(z)}}{\mathbb{E}[\mathrm{e}^{-Y_d(z)}]}$$

where

$$Y_d(z) = \sum_{k\geqslant 1}\frac{z^k}{kd^{k/2}}\sum_{\ell|k}\ell(\Lambda_\ell - \mathbb{E}[\Lambda_\ell]).$$

Theorem 2.2.7 can be seen as an analog of Theorem 2.2.3. In both cases, the characteristic polynomial converges to the exponential of a random series, whose coefficients are Gaussian for circular ensembles and Poisson for permutation matrices.

The appearance of Poisson variables in the limit for permutation matrices arises from known results on the convergence of the cycle process. Indeed, for $|z| < 1$, the log-characteristic polynomial $\log p_n$ can be written as a sum over random cycle counts

$$\log p_n(z) = \sum_{k=1}^{n} C_k^{(n)} \log(1 - z^k).$$

The joint distribution of cycles counts $\left(C_k^{(n)}\right)_{1 \leqslant k \leqslant n}$ for uniform permutation matrices is known to converge to Poisson random variables, as shown in [SL66]. This result was extended in [CLZ24] to sums of a fixed number of independent, uniform permutation matrices. For a permutation matrix A_n with cycle decomposition $\left(C_k^{(n)}\right)_{1 \leqslant k \leqslant n}$, there is also an explicit relation between the trace of powers and cycle counts:

$$\operatorname{Tr}\left[A_n^k\right] = \sum_{\ell | k} \ell C_\ell^{(n)}.$$

Together with the finite dimensional convergence of $\left(C_\ell^{(n)}\right)_{\ell \geqslant 1}$ to Poisson random variables, one expects the limiting function for the characteristic polynomial of random uniform permutation matrices to be given by the exponential of a Poisson series which is precisely the result of Theorem 2.2.7.

In their work [CLZ24], the authors raise the question of whether their result can be extended to more general measures on permutations, particularly the Ewens distribution (1.1.12). In Chapter 5, we address this question by establishing the convergence of the characteristic polynomial for a measure that generalizes the Ewens distribution, namely, the generalized Ewens distribution, introduced by Nikeghbali and Zeindler in [NZ13].

2.3 Study of outliers

We present here a series of results on the convergence of characteristic polynomials, originally motivated by the study of outliers. It was noticed by Basak and Zeitouni [BZ20] that perturbations of Toeplitz matrices, which are non-normal random matrices, have some eigenvalues that deviate from the global behavior dictated by the convergence of the empirical eigenvalue distribution.
Motivated by techniques used for Toeplitz matrices, Bordenave, Chafaï and García [BCG22] proved the convergence of the characteristic polynomial of Girko matrices under a universal second order moment condition. Their result led to the convergence of the spectral radius for such matrices as stated in Theorem 1.4.4.
This approach has since been applied to other matrix models. For i.i.d. but non-centered entries with Bernoulli distribution, Coste [Cos23] established the convergence of the characteristic polynomial to a limiting random function. The same kind of convergence was obtained by Coste, Lambert and Zhu [CLZ24] for sums of random uniform permutation matrices and for Circular β-Ensembles by Najnudel, Paquette, Simm [NPS23], as presented in Theorem 2.2.7 and Theorem 2.2.3 respectively.
The results presented in this section concern various matrix models, namely Toeplitz,

Girko and Bernoulli which are related by the convergence of their characteristic polynomial toward limiting functions involving either Gaussian analytic functions or, in the case of permutation models, Poisson series.

2.3.1 Toeplitz matrices

We begin with the seminal work of Basak and Zeitouni [BZ20], from which this section borrows from, on outliers of randomly perturbed Toeplitz matrices which inspired the developments presented in Sections 2.3.2 and 2.3.3.

Let d_1, d_2 be positive integers and let $(a_k)_{-d_2 \leqslant k \leqslant d_1}$ be the coefficients of a Laurent polynomial

$$\mathbf{a}(z) = \sum_{k=-d_2}^{d_1} a_k z^k.$$

Definition 2.3.1 (Toeplitz matrix)**.** For $n \geqslant \max(d_1, d_2)$, the $n \times n$ *Toeplitz matrix with symbol* $\mathbf{a}$ is defined as

$$T_n(\mathbf{a}) \coloneqq \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & 0 \\ a_{-1} & a_0 & a_1 & \ddots & \vdots \\ a_{-2} & a_{-1} & a_0 & \ddots & a_2 \\ \vdots & \ddots & \ddots & \ddots & a_1 \\ 0 & \cdots & a_{-2} & a_{-1} & a_0 \end{pmatrix}.$$

For convenience we write T_n for $T_n(\mathbf{a})$. The matrix T_n can be seen as a finite-dimensional approximation of the Toeplitz operator $T = T(a) : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ defined by

$$T(x) \coloneqq \left(\sum_{k=-d_2}^{d_1} a_k x_{k+n}. \right)_{n \geqslant 0}, \ x = (x_n)_{n \geqslant 0}.$$

Let $\mathrm{Spec}(T)$ denote the spectrum of the operator T. It is known that the empirical eigenvalue distribution of T_n converges weakly in probability to the law of $a(U)$, where U is a random variable uniformly distributed on $\mathbb{S}^1$.
The results of [BZ20] concern random perturbation of T_n. Under suitable conditions on the perturbation Δ_n, no eigenvalue of $A_n = T_n + \Delta_n$ lies outside of a small neighborhood of the limiting support $\mathbf{a}(\mathbb{S}^1)$, a property known as the *spectral stability.*

Theorem 2.3.2 (Spectral stability for $T(a)$, [BZ20])**.** *Let $\Delta_n = n^{-\gamma} E_n$ for some $\gamma > \frac{1}{2}$, where $E_n \in \mathcal{M}_n(\mathbb{C})$ is a Girko matrix with centered entries of unit variance. Let μ_n be the empirical eigenvalue distribution of $A_n = T_n + \Delta_n$. Then, for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left[\mu_n\left(\{z \in \mathbb{C} \mid \mathrm{dist}(z, \mathrm{Spec}(T(\mathbf{a}))) > \epsilon\}\right) = 0\right] = 1.$$

In addition, the convergence of the eigenvalue point process outside $\mathrm{Spec}(T(\mathbf{a}))$ was also established. Let $\mathcal{S}_0 = \mathbb{C} \setminus \mathrm{Spec}(T(\mathbf{a}))$ be the region outside the limiting support for the eigenvalues. Let $\mathrm{Spec}_n = \{z \in \mathbb{C} \mid \det(zI_n - A_n) = 0\}$ be the spectrum of $A_n = T_n(a) + \Delta_n$ and consider the point process

$$\Xi_n = \sum_{z \in \mathcal{S}_0 \cap \mathrm{Spec}_n} \delta_z.$$

It was shown in [BZ20, Theorem 1.11], that under further assumptions on the perturbation matrix Δ_n, the point process Ξ_n converges weakly to a limiting point process described as the zeros of a random field $(F(z))_{z\in\mathcal{S}_0}$. Explicit expressions for this limiting field are known for particular instances of the symbol $\mathbf{a}$. In the case where $\mathbf{a}(z) = z$ and where E_n is a Ginibre matrix, the limit is given by the *hyperbolic Gaussian analytic function*:

$$F(z) = \sum_{k\geqslant 0} z^k X_k \sqrt{k+1} \tag{2.3.1}$$

where $(X_k)_{k\geqslant 0}$ are i.i.d. standard complex Gaussians. The function appearing in (2.3.1) is another example of a Gaussian analytic function as in Definition 2.2.2.

2.3.2 Girko matrices

We now turn to results on the characteristic polynomials of i.i.d. matrices based on the work [BCG22]. In particular, this will give a proof of the previously stated convergence of the spectral radius of Girko matrices in Theorem 1.4.4. In this section, A_n denotes a Girko matrix with entries satisfying

$$\mathbb{E}[a_{ij}] = 0 \text{ and } \mathbb{E}[|a_{ij}|^2] = 1.$$

The main result of [BCG22] is the convergence in law, for the topology of local uniform convergence, of the scaled characteristic polynomial

$$p_n(z) \coloneqq \det\left(I_n - z\frac{A_n}{\sqrt{n}}\right)$$

inside the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem 2.3.3 (Convergence of characteristic polynomial for Girko matrices, [BCG22])**.** *We have the convergence in law*

$$p_n \underset{n\to\infty}{\to} \kappa \mathrm{e}^{-F}$$

where

$$\kappa(z) = \sqrt{1 - z^2\mathbb{E}[a_{11}^2]},\ F(z) = \sum_{k\geqslant 1} X_k \frac{z^k}{\sqrt{k}}$$

with $(X_k)_{k\geqslant 1}$ a family of independent, complex Gaussians such that

$$\mathbb{E}[X_k] = 0,\ \mathbb{E}[|X_k|^2] = 1 \text{ and } \mathbb{E}[X_k^2] = \mathbb{E}[a_{11}^2]^k.$$

Theorem 2.3.3 implies the convergence of the spectral radius ρ_n of $\frac{1}{\sqrt{n}}A_n$. Indeed, from the continuous mapping theorem, for every $r \in (0, 1)$,

$$\mathbb{P}\left[\rho_n < \frac{1}{r}\right] = \mathbb{P}\left[\inf_{|z|\leqslant r} |p_n(z)| > 0\right] \xrightarrow[n\to\infty]{} \mathbb{P}\left[\inf_{|z|\leqslant r} |\kappa(z)\mathrm{e}^{-F(z)}| > 0\right] = 1.$$

The expression of the limit as the exponential of a Gaussian analytic function connects with the case of unitary matrices as the same expression of the exponential of a Gaussian analytic function appears for the limiting random function. The convergence of the characteristic polynomial for Girko matrices with variance profile can be found in [HL25]. In this case, the limiting Gaussian analytic function depends on the variance profile.

2.3.3 Bernoulli matrices

We now present a result by Coste [Cos23] concerning the convergence of the characteristic polynomial for matrices with i.i.d. Bernoulli entries. The main difference compared to the results of [BCG22] is that Bernoulli random variables are non-centered.
In this section, $A_n = (a_{ij})_{1\leqslant i,j\leqslant n}$ is a $n\times n$ matrix with entries a_{ij} following the Bernoulli distribution of parameter $\frac{d_n}{n}$, denoted by $a_{ij}\sim \mathcal{B}(\frac{d_n}{n})$ for some sequence $(d_n)_{n\geqslant 1}$ of positive numbers. This corresponds to the adjacency matrix of a directed Erdös-Rényi graph with average in and out degrees equal to d_n, which explains the scaling $\frac{d_n}{n}$.

The empirical eigenvalue distribution of A_n has been studied by Rudelson, Tikhomirov [RT19] and Basak [BR19]. The limiting distribution depends on the asymptotic behavior of the sequence $(d_n)_{n\geqslant 1}$. In particular,

- if $d_n \to \infty$, the empirical eigenvalue distribution converges almost surely to the circular law as in Theorem 1.3.12.
- If $d_n = d$, called the *sparse regime*, no closed expression of the limit eigenvalue distribution is known. However, results on eigenvalue localization are established in [BCN23].

In the sparse regime, Coste [Cos23] established the convergence of the characteristic polynomial, thereby providing a new proof of the localization results in [BCN23]. We state this convergence result below. For $r\in(0,1)$, we denote by $\mathbb{D}_r = \{z\in\mathbb{C} : |z| < r\}$ the open disk of radius r.

Theorem 2.3.4 (Convergence of characteristic polynomial of sparse Bernoulli matrices, [Cos23])**.** *Let $d>0$ and let A_n be a matrix having i.i.d. Bernoulli entries with parameter $\frac{d}{n}$. Then, as $n\to\infty$, we have the convergence in law, for the topology of local uniform convergence in $\mathcal{H}(\mathbb{D}_{1/\sqrt{d}})$,*

$$p_n \longrightarrow F$$

where, for z close enough to the origin,

$$F(z) = \prod_{k\geqslant 1}\left(1 - z^k\right)^{Y_k},$$

with $(Y_k)_{k\geqslant 1}$ being independent Poisson random variables with parameters $(\frac{d^k}{k})_{k\geqslant 1}$.

More refined expressions of the limiting function F depending on the value of d can be found in [Cos23, Theorem 2.7], cases $(i)-(iii)$.

The limit expression is reminiscent of the one obtained for the characteristic polynomial of permutation matrices in Theorem 2.2.7. In both cases, the limit can be written as the exponential of a random series with Poisson distributed coefficients. In analogy with the holomorphic multiplicative chaos introduced in [NPS23], Coste referred to the limiting function of Theorem 2.3.4 as the *Poisson multiplicative chaos*. Poisson series also appear as limits of the characteristic polynomial of Ewens permutation matrices, see chapter 5.

2.4 Contributions to the subject

We state our contributions on the convergence of characteristic polynomials for two integrable models presented in Sections 2.4.1 and Sections 2.4.2 and corresponding to the articles [FG23] and [Fra25] which are the subjects of Chapters 4 and 5 respectively.

2.4.1 Characteristic polynomial of Gaussian elliptic matrices

The random matrices that we consider in this section are sampled from the complex elliptic Ginibre Ensemble introduced by Girko in [Gir86]. This model is parametrized by $t \in [0,1]$ and interpolates between the Ginibre Ensemble of Definition 1.1.5 and the Gaussian Unitary Ensemble of Definition 1.1.8 for $t = 0$ and $t = 1$ respectively. Its law is the one of a random matrix given by the following construction.

Consider X_n and Y_n independent random matrices sampled from the Gaussian Unitary Ensemble of size $n \geqslant 1$. The law of the elliptic Ginibre Ensemble at $t \in [0,1]$ is the law of the matrix

$$A_{n,t} = \sqrt{\frac{1+t}{2}} X_n + i\sqrt{\frac{1-t}{2}} Y_n, \tag{2.4.1}$$

where i is the imaginary unit. Equivalently, $A_{n,t}$ has a law proportional to

$$\exp\left(-\frac{1}{1-t^2}\mathrm{Tr}\left[M^*M - \frac{t}{2}\left(M^2 + (M^*)^2\right)\right]\right) \mathrm{d}M, \tag{2.4.2}$$

where $\mathrm{d}M = \prod_{1\leqslant i,j\leqslant n} \mathrm{d}M_{ij}$ is the product Lebesgue measure on the entries of the matrix. The limiting eigenvalue distribution has been proved by Girko [Gir86] to be the uniform law on the ellipse

$$\mathcal{E}_t := \left\{ x + iy \in \mathbb{C} \mid \left(\frac{x}{1+t}\right)^2 + \left(\frac{y}{1-t}\right)^2 \leqslant 1 \right\}.$$

Let us define $f_{n,t} : \mathbb{D} \to \mathbb{C}$ as the *normalised characteristic polynomial* of $A_{n,t}$,

$$f_{n,t}(z) := \det\left(1 + tz^2 - \frac{z}{\sqrt{n}} A_{n,t}\right) \mathrm{e}^{-\frac{ntz^2}{2}}. \tag{2.4.3}$$

We endow the space of holomorphic functions on $\mathbb{D}$ with the topology of uniform convergence on compact sets. Our main result is the following convergence.

Theorem 2.4.1 (Convergence of the normalised characteristic polynomial)**.** *We have the convergence in law, for the topology of local uniform convergence,*

$$f_{n,t} \xrightarrow[n\to\infty]{\text{law}} \exp(-F_t)$$

where F_t is the Gaussian holomorphic function on $\mathbb{D}$ defined by

$$F_t(z) := \sum_{k\geqslant 1} X_k \frac{z^k}{\sqrt{k}} \tag{2.4.4}$$

for a family $(X_k)_{k\geqslant 1}$ of independent Gaussian random variables on $\mathbb{C}$ satisfying

$$\mathbb{E}[X_k] = 0,\ \mathbb{E}[X_k^2] = t^k \text{ and } \mathbb{E}[|X_k|^2] = 1.$$

In particular, for $t = 1$, Theorem 2.4.1 shows that the characteristic polynomial of GUE matrices, suitably normalized, converges to a random holomorphic function. From Theorem 2.4.1, we derive the absence of outliers which is the elliptic analog of the convergence of the spectral radius of Girko matrices, see Theorem 1.4.4.

Corollary 2.4.2 (Lack of outliers)**.** *Let $C \subset \mathbb{C}$ be a closed set disjoint from $\mathcal{E}_t$. Then,*

$$N_n(C) := \#\left\{ i \in [n] : \frac{\lambda_i}{\sqrt{n}} \in C \right\} \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{2.4.5}$$

We expect an analogue of Theorem 2.4.1 to hold in a much more general setting as conjectured in [BCG22], see Section 2.5.1. The limit would only depend on some of the first four moments of the coefficients of the random matrix. A glimpse of this universality can be seen, for instance, when calculating the expected value of the characteristic polynomial. This depends only on $t = \mathbb{E}[a_{12}a_{21}]$ and we have the following convergence for the average characteristic polynomial of elliptic matrices.

Theorem 2.4.3 (Average characteristic polynomial)**.** *For each n, let $A_{n,t} = (a_{ij}, 1 \leqslant i, j \leqslant n)$ be a random matrix such that $\{(a_{ij}, a_{ji}), 1 \leqslant i < j \leqslant n\}$ are i.i.d. centered pairs which are independent of the i.i.d. centered family $\{a_{ii}, 1 \leqslant i \leqslant n\}$ with $\mathbb{E}[|a_{ij}|^2] < \infty$ for all $1 \leqslant i, j \leqslant n$ and $\mathbb{E}[a_{12}a_{21}] = t \in [0, 1]$. Then, for z uniformly in $\mathbb{D}$,*

$$\lim_{n\to+\infty} \mathbb{E}\left[\det\left(1 + tz^2 - \frac{z}{\sqrt{n}} A_{n,t}\right) \mathrm{e}^{-\frac{ntz^2}{2}}\right] = \frac{1}{\sqrt{1 - tz^2}}. \tag{2.4.6}$$

2.4.2 Characteristic polynomial of generalized Ewens random matrices

In this section, we consider permutation matrices sampled from the *generalized Ewens distribution*, introduced by Nikeghbali and Zeindler [NZ13], which generalizes the Ewens distribution (1.1.12). Recall that for a permutation $\sigma \in S_n$ and $k \geqslant 1$, $C_k(\sigma)$ is the number of cycles of σ with length k.

Definition 2.4.4 (Generalized Ewens measure, [NZ13])**.** Let $\Theta = (\theta_k)_{k\geqslant 1}$ be a sequence of positive real numbers. For $n \geqslant 1$, the *generalized Ewens measure* is the probability measure $\mathrm{d}\mathbb{P}_n^\Theta$ on S_n defined by

$$\mathrm{d}\mathbb{P}_n^\Theta[\sigma] \coloneqq \frac{1}{n! h_n^\Theta} \prod_{k=1}^{n} \theta_k^{C_k(\sigma)}. \tag{2.4.7}$$

From the sequence $\Theta = (\theta_k)_{k\geqslant 1}$, one defines the formal power series as in [NZ13],

$$g_\Theta(z) \coloneqq \sum_{k\geqslant 1} \frac{\theta_k}{k} z^k \text{ and } G_\Theta(z) \coloneqq \exp(g_\Theta(z)) \tag{2.4.8}$$

For $n \geqslant 1$ and $\Theta = (\theta_k)_{k\geqslant 1}$ as above, we consider A_n the random matrix associated to a permutation σ sampled from (2.4.7). Let us consider the characteristic polynomial

$$p_n(z) = \det(1 - zA_n) \tag{2.4.9}$$

inside the unit disk $z \in \mathbb{D} = \{x \in \mathbb{C} : |x| < 1\}$. Let us denote by $\mathcal{H}(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$ endowed with the topology of convergence on compact subsets of $\mathbb{D}$. Our main result is the convergence of p_n as a random variable in $\mathcal{H}(\mathbb{D})$ in law towards a limit function $F \in \mathcal{H}(\mathbb{D})$. The above convergence holds for parameters Θ such that the generating series g_Θ satisfies some conditions that we now define which is an adaptation of a definition given in Section 5.2.1 of [Hwa94]. One can also find it as Definition 2.9 in [Hug+13] or Definition 2.8 in [NZ13].

Definition 2.4.5 (Logarithmic class function)**.** A function g is said to be in $F(r, \gamma, K)$ for $r > 0$, $\gamma \geqslant 0$ and $K \in \mathbb{C}$ if

- There exists $R > r$ and $\phi \in (0, \pi/2)$ such that g is holomorphic in $\Delta(r, R, \phi) \setminus \{r\}$ where $\Delta(r, R, \phi) = \{z \in \mathbb{C} : |z| \leqslant R, |\arg(z - r)| \geqslant \phi\}$.
- As $z \to r$, $g(z) = -\gamma \log(1 - z/r) + K + O(z - r)$.

In the case of the Ewens measure of parameter θ, we have $g_\Theta(z) = -\theta \log(1-z)$ so that $g_\Theta \in F(1, \theta, 0)$. Note that if $\gamma > 0$, the parameter r is unique.

Our main result is Theorem 2.4.6 which gives the convergence of the characteristic polynomial towards a limit function for sequences Θ such that g is of logarithmic class.

Theorem 2.4.6 (Convergence of the characteristic polynomial)**.** *Let $\Theta = (\theta_k)_{k\geqslant 1}$ be a sequence of positive real numbers such that $g_\Theta \in F(r, \gamma, K)$ for $r > 0$ and $\gamma > 0$. We have the convergence in law, for the topology of local uniform convergence in $\mathbb{D}$*

$$p_n \xrightarrow[n\to\infty]{\text{law}} F, \tag{2.4.10}$$

where

$$F(z) = \exp\left(-\sum_{k\geqslant 1} \frac{z^k}{k} X_k\right), \quad X_k = \sum_{\ell | k} \ell Y_\ell, \tag{2.4.11}$$

with $(Y_\ell)_{\ell\geqslant 1}$ independent Poisson random variables with parameter $\frac{\theta_\ell}{\ell} r^\ell$.

The previous theorem gives in particular the convergence of the characteristic polynomial for Ewens permutation matrices. Indeed, for constant θ, the function $g_\Theta \in F(1, \theta, 0)$ so that p_n converges towards the limit function as conjectured in [CLZ24].

2.4.3 Summary of presented limits for characteristic polynomials

Matrix model	Scaling	Limit function
CβE [NPS23]	$\det(I_n - zA_n)$ $z \in \mathbb{D}$	Holomorphic Multiplicative Chaos $\exp\left(\sqrt{\frac{2}{\beta}}\sum_{k\geqslant 1}\frac{z^k}{\sqrt{k}}X_k\right)$ $X_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$.
Haar unitary [CNN17]	$\xi_n(z) = \frac{\det\left(I_n - \mathrm{e}^{\frac{2i\pi z}{n}}A_n^*\right)}{\det(I_n - A_n^*)}$ $z \in \mathbb{C}$	$\xi_\infty(z) = \mathrm{e}^{i\pi z}\prod_{k\in\mathbb{Z}}\left(1 - \frac{z}{y_k}\right)$ $(y_k) \sim$ *Sine process.*
Sum of d uniform permutations [CLZ24]	$\frac{1}{\sqrt{d}}p_n\left(\frac{z}{\sqrt{d}}\right)$ $z \in \mathbb{D}$	Poisson Multiplicative Function $\left(z - \frac{1}{\sqrt{d}}\right)\frac{\mathrm{e}^{-Y_d(z)}}{\mathbb{E}[\mathrm{e}^{-Y_d(z)}]}$ $Y_d(z) = \sum_{k\geqslant 1}\frac{z^k}{kd^{k/2}}\sum_{\ell\mid k}\ell(\Lambda_\ell - \mathbb{E}[\Lambda_\ell])$ $(\Lambda_\ell) \sim \mathcal{P}\left(\frac{d^\ell}{\ell}\right)$.
Ewens permutations [Fra25] $\boldsymbol{\Theta} = (\theta_\ell)_{\ell\geqslant\mathbf{1}}$ $\mathbf{g}_{\boldsymbol{\Theta}} \in \mathbf{F}(\mathbf{r},\gamma,\mathbf{K})$	$\det(\mathbf{I_n} - \mathbf{zA_n})$ $\mathbf{z} \in \mathbb{D}$	**Poisson Multiplicative Function** $\exp\left(-\sum_{\mathbf{k}\geqslant\mathbf{1}}\frac{\mathbf{z^k}}{\mathbf{k}}\mathbf{X_k}\right)$ $\mathbf{X_k} = \sum_{\ell\mid\mathbf{k}}\ell\mathbf{Y}_\ell,\ \mathbf{Y}_\ell \sim \mathcal{P}\left(\frac{\mathbf{r}^\ell\theta_\ell}{\ell}\right)$.
Girko [BCG22] $\mathbb{E}[a_{ij}] = 0$ $\mathbb{E}[\lvert a_{ij}\rvert^2] = 1$	$\det\left(I_n - z\frac{A_n}{\sqrt{n}}\right)$ $z \in \mathbb{D}$	Gaussian Multiplicative Chaos $\sqrt{1 - z^2\mathbb{E}[a_{11}^2]}\exp\left(-\sum_{k\geqslant 1}\frac{z^k}{\sqrt{k}}X_k\right)$ $(X_k)_{k\geqslant 1}$ independent, complex Gaussians $\mathbb{E}[X_k] = 0,\ \mathbb{E}[\lvert X_k\rvert^2] = 1, \mathbb{E}[X_k^2] = \mathbb{E}[a_{11}^2]^k$.
Elliptic Ginibre [FG23] $\mathbf{t} \in [\mathbf{0},\mathbf{1}]$	$\det\left(\mathbf{1} + \mathbf{tz^2} - \frac{\mathbf{z}}{\sqrt{\mathbf{n}}}\mathbf{A_{n,t}}\right)\mathrm{e}^{-\frac{\mathbf{ntz^2}}{\mathbf{2}}}$ $\mathbf{z} \in \mathbb{D}$	**Gaussian Multiplicative Chaos** $\exp\left(-\sum_{\mathbf{k}\geqslant\mathbf{1}}\frac{\mathbf{z^k}}{\sqrt{\mathbf{k}}}\mathbf{X_k}\right)$ $(\mathbf{X_k})_{\mathbf{k}\geqslant\mathbf{1}}$ **independent, complex Gaussians** $\mathbb{E}[\mathbf{X_k}] = \mathbf{0},\ \mathbb{E}[\mathbf{X_k^2}] = \mathbf{t^k},\ \mathbb{E}[\lvert\mathbf{X_k}\rvert^{\mathbf{2}}] = \mathbf{1}$.
Girko Bernoulli [Cos23] $a_{ij} \sim \mathcal{B}(\frac{d}{n})$	$\det(I_n - zA_n)$ $z \in \mathbb{D}$	Poisson Multiplicative Function $\prod_{k\geqslant 1}(1 - z^k)^{Y_k}$ $Y_k \sim \mathcal{P}\left(\frac{d^k}{k}\right)$.

2.5 Open questions

2.5.1 Minimal moment condition and universality

As conjectured in [BCG22], the convergence in Theorem 2.4.1 of the normalised characteristic polynomial is believed to hold under the minimal moment condition

$$\mathbb{E}\left[|a_{12}a_{21}|^2\right] < \infty$$

on the entries $(a_{ij})_{i,j\geqslant 1}$, which gives a condition of a fourth order moment for Wigner matrices and second order moment for Girko matrices. The context adapted to this conjecture is the one of elliptic random matrices [NO15, Definition 1.3]. This model was introduced by Girko in [Gir86] and [Gir95]. A version of this model consists of the following matrices. Consider a family $(a_{ij})_{i,j\geqslant 1}$ of square-integrable centered random variables such that $\{(a_{ij}, a_{ji}) : i < j\} \cup \{a_{ii} : i \geqslant 1\}$ is an independent family of random elements and whose law is invariant under any permutation of the indices or, equivalently, the law of (a_{ij}, a_{ji}) coincides with the law of $(a_{i'j'}, a_{j'i'})$ whenever $|\{i,j\}| = |\{i',j'\}|$. If

$$\mathbb{E}[|a_{12}|^2] = 1 \quad \text{and} \quad \mathbb{E}[a_{12}a_{21}] = t,$$

the matrix $A_n = (a_{ij})_{1\leqslant i,j\leqslant n}$ is said to be t-Girko. The convergence of the average eigenvalue distribution towards the uniform distribution on the ellipse has been proved under different conditions on the variables, see [NO15; OR14; Nau13]. We expect the following version of Theorem 2.4.1 to hold for the general t-Girko matrices described above. Denoting $\tau = \mathbb{E}[a_{12}^2]$, $s = \mathbb{E}[a_{11}^2] - t - \tau$ and $q = \mathbb{E}[(a_{12}a_{21} - t)^2] - t^2 - \tau^2$, the limit of $\det\left(1 + tz^2 - z\frac{A_{n,t}}{\sqrt{n}}\right)\exp(-ntz^2/2)$ is expected to be given by

$$\sqrt{1-\tau z^2}\mathrm{e}^{-sz^2/2}\mathrm{e}^{-qz^4/4}\mathrm{e}^{-\sum_{k\geqslant 1} Y_k \frac{z^k}{\sqrt{k}}}$$

where $(Y_k)_{k\geqslant 1}$ are independent centered complex Gaussians such that Y_1 has the same variance as a_{11}, Y_2 has the same variance as $a_{12}a_{21}$ and, for $k \geqslant 3$, the variance of Y_k is the sum of the k-th power of the variance of a_{12} and the k-th power of the covariance of a_{12} and a_{21} or, somewhat more explicitly, $\mathbb{E}[Y_k^2] = \mathbb{E}[a_{12}^2]^k + \mathbb{E}[a_{12}a_{21}]^k = \tau^k + t^k$ and $\mathbb{E}[|Y_k|^2] = \mathbb{E}[|a_{12}|^2]^k + \mathbb{E}[a_{12}\overline{a_{21}}]^k = 1 + \mathbb{E}[a_{12}\overline{a_{21}}]^k$.

2.5.2 Matrices with entries in $\{0,1\}$

As presented in Section 2.3.3, a convergence of the reciprocal characteristic polynomial for matrices with independent Bernoulli entries with non-zero expectation has been proved in [Cos23]. The limiting random holomorphic function can be expressed using Poisson random variables, see Theorem 2.3.4. One could ask for an analogue of the elliptic Ginibre Ensemble for such matrices and for the convergence of its characteristic polynomial.

2.5.3 Determinantal Coulomb gases

The convergence of Theorem 2.4.1 can be thought of as a first step towards the convergence of the characteristic polynomial outside the support of the equilibrium measure for general elliptic random matrices. Nevertheless, we could have followed a different path, which is to look the Elliptic Ginibre Ensembles as a particular case of a determinantal Coulomb gas. In this vein, it may be possible to show the convergence of the traces by adapting results from [AHM15] and to show tightness of the characteristic polynomial outside the support of the equilibrium measure for more general determinantal Coulomb gases by using, for instance, the results from [AC23].

2.5.4 Characteristic polynomial in the bulk

Theorem 2.4.6 shows the convergence of the characteristic polynomial outside of the support of the limiting eigenvalue distribution. One could ask for a similar study inside the region where the eigenvalues are, that is, for the limiting distribution of $\log p_n(z)$ for z inside the asymptotic support. Expanding the logarithm gives

$$\log p_n(z) = \sum_{k=1}^{n} \log(1 - z\lambda_{k,n}) = n \int \log(1 - zu)\mu_n(\mathrm{d}u)$$

so that the asymptotic analysis can be viewed as a central limit theorem for the log statistic. Limits for fluctuations of $\log |p_n(z)|$ for Ginibre matrices inside the unit disk have been established to be Gaussian [RV07], and the limiting field inside the bulk is the Gaussian free field. As suggested by the results of Webb and Wong [WW19], the scaling would be different compared to the outside region. Central limit theorems for linear statistics were proven by Rider and Silverstein for general complex Girko matrices [RS06] and by [CES21] for the case of real entries, with regularity assumptions on test functions. In another universal direction, fluctuations results were established for linear statistics of Coulomb gases [LS18; Bau+19] where the limiting field is the Gaussian free field.

2.5.5 Fluctuations of real parts

One could also study the fluctuations of real parts of eigenvalues of matrices from the elliptic Ginibre Ensemble. In the case of the GUE, it is known by the work of Gustavsson [Gus05] that the k-th eigenvalue has asymptotic Gaussian fluctuations around its expected location in the semi-circle, both in the bulk when $\frac{k}{n} \to a \in (0,1)$ and in the edge when $k \to \infty$ and $\frac{k}{n} \to 0$. The proof relies on a result of Costin, Lebowitz [CL95] and Soshnikov [Sos00b] giving Gaussian fluctuations for the number of points of a determinantal point process which lie in some interval. Since eigenvalues of the elliptic Ginibre Ensemble form a determinantal point process and results of [ADM23] provide asymptotics for the associated kernel, one could aim at deriving Gaussian fluctuations for their real parts by using these techniques.

Chapter 3

Horn problems

This chapter provides an overview of research initiated by Horn [Hor62] which arose from a question of Weyl [Wey12] on the spectrum of a sum of Hermitian matrices in 1912. This initial problem is known as the additive or Hermitian Horn Problem. Since its original formulation, determining exact conditions that characterize the spectrum of a sum of Hermitian matrices led to many developments using techniques from representation theory [Kly98], algebraic and symplectic geometry [Knu00], combinatorics [KT01] and probability theory [CMZ19]. The complete solution to the original problem was only given in the late 1990s, notably through the contributions of Klyachko [Kly98], Totaro [Tot94], Knutson and Tao [KT99]. Section 3.1 presents the main developments starting from the work of Horn.

From the initial additive problem, new versions were studied for eigenvalues of operations on subgroups of matrices. One particular case is to characterise the eigenvalues of products of unitary matrices, also known as the unitary, or multiplicative Horn problem. This unitary version is the subject of Section 3.2. As with the additive problem, the multiplicative version turns out to be related to notions coming from representation theory, geometry and mathematical physics. In Section 3.3, we present our results on a probabilistic version of the unitary Horn problem, corresponding to the articles [FT24] and [Fra24] respectively presented in Chapters 6 and 7 of this thesis.

3.1 Sum of Hermitian matrices

This section is devoted to the additive version of the Horn problem. It is inspired by the surveys of Fulton [Ful00] and of Knutson and Tao [KT01]. We begin in Section 3.1.1 by giving the historical perspective that led to a system of inequalities which eigenvalues of a sum must satisfy and which turn out be to sufficient. Section 3.1.2 reinterprets these inequalities from a geometric point of view and introduces the Littlewood-Richardson coefficients which are central in the additive Horn problem. Section 3.1.3 presents combinatorial results on Littlewood-Richardson coefficients based on the work of Knutson and Tao. Beyond the original deterministic problem, one can consider a probabilistic version: given two random Hermitian matrices what is the density of eigenvalues of their sum. This probabilistic framework is presented in Section 3.1.4.

3.1.1 System of inequalities for eigenvalues

Let $n \geqslant 1$ be a fixed integer. In 1912, Weyl [Wey12] asked the following question.

Given two Hermitian matrices, what eigenvalues can arise for their sum ?

Let us consider Hermitian matrices A and B with fixed respective eigenvalues

$$\alpha = (\alpha_1 \geqslant \cdots \geqslant \alpha_n) \text{ and } \beta = (\beta_1 \geqslant \cdots \geqslant \beta_n) \ .$$

This is equivalent to saying that A and B are in the respective orbits $\mathcal{O}^H(\alpha)$ and $\mathcal{O}^H(\beta)$ where for $\theta \in \mathbb{R}^n/S_n$,

$$\mathcal{O}^H(\theta) := \{U \operatorname{Diag}(\theta_1, \ldots, \theta_n) U^*, U \in \mathrm{U}(n)\}.$$

Let $C = A + B$ be their sum, which is also a Hermitian matrix and denote its eigenvalues by

$$\gamma = (\gamma_1 \geqslant \cdots \geqslant \gamma_n).$$

The question can be formulated as

What is the relation between γ and α, β ?

A first equality relating α, β and γ is obtained by taking the trace which gives

$$\sum_{k=1}^{n} \alpha_k + \sum_{k=1}^{n} \beta_k = \sum_{k=1}^{n} \gamma_k \ . \tag{3.1.1}$$

Thus, the eigenvalues γ are located in a hyperplane of $\mathbb{R}^n$. Recall that for a Hermitian matrix $A \in \mathcal{M}_n$,

$$\alpha_1 = \max_{x:||x||=1} \langle x, Ax \rangle$$

where $\langle x, y \rangle$ denotes the Hermitian scalar product. Thus

$$\gamma_1 \leqslant \alpha_1 + \beta_1.$$

This inequality can be generalized using min-max formulas for eigenvalues of Hermitian matrices. For $1 \leqslant k \leqslant n$, let us denote by

$$\mathbb{G}r(k, n) := \{V \in \mathbb{C}^n \mid \dim(V) = k\}$$

the set of k-dimensional subspaces of $\mathbb{C}^n$, called the *Grassmannian* of k-dimensional subspaces. Any eigenvalue can then be expressed via a min–max formula thanks to the results of Courant [Cou20] and Fischer [Fis05].

Lemma 3.1.1 (Minmax formulation, [Cou20; Fis05])**.** *Let A be a Hermitian matrix of size $n \geqslant 1$. For any $1 \leqslant k \leqslant n$,*

$$\alpha_k = \max_{V \in \mathbb{G}r(k,n)} \min_{\substack{x \in V \\ ||x||=1}} \langle x, Ax \rangle = \min_{V \in \mathbb{G}r(n+1-k,n)} \max_{\substack{x \in V \\ ||x||=1}} \langle x, Ax \rangle. \tag{3.1.2}$$

For $1 \leqslant r \leqslant n$, let us denote by

$$\mathcal{P}_n^r := \{I = (i_1, \ldots, i_r) \mid 1 \leqslant i_1 < i_2 \cdots < i_r \leqslant n\}$$

the set of ordered r-tuples distinct elements of $[n]$. From Lemma 3.1.1, one can obtain the following result.

Lemma 3.1.2 (Eigenspace)**.** *Let $x_1, \dots, x_n$ be an orthonormal basis of eigenvectors of A associated to its eigenvalues $\alpha_1, \dots, \alpha_n$. Let $r \in [n]$ and let $I = (i_1, \dots, i_r) \in \mathcal{P}_n^r$. Let $V = \mathrm{Vect}(x_{i_1}, \dots, x_{i_r})$ be the subspace generated by eigenvectors indexed by I. Then,*

$$\alpha_{i_1} = \max_{\substack{x \in V \\ ||x||=1}} \langle x, Ax \rangle, \quad \alpha_{i_r} = \min_{\substack{x \in V \\ ||x||=1}} \langle x, Ax \rangle.$$

Using Lemma 3.1.2, one derives the Weyl inequalities that can be found in [Wey12].

Lemma 3.1.3 (Weyl inequalities, [Wey12])**.** *For $1 \leqslant i, j \leqslant n$ such that $i + j - 1 \leqslant n$,*

$$\gamma_{i+j-1} \leqslant \alpha_i + \beta_j. \tag{3.1.3}$$

For $n = 2$, the inequalities (3.1.3) together with the trace condition (3.1.1) are both necessary and sufficient for the existence of three Hermitian matrices with eigenvalues α, β, γ related by $C = A + B$.
More inequalities are needed to fully characterise the eigenvalues of the sum. The general form of these inequalities is given by

$$\sum_{k \in K} \gamma_k \leqslant \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \ , \tag{IJK}$$

and is parametrized by triples $(I, J, K) \in (\mathcal{P}_n^r)^3$ for $r < n$. For $r < n$, define

$$U_n^r := \left\{ (I, J, K) \in (\mathcal{P}_n^r)^3 \mid \sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + \frac{r(r-1)}{2} \right\}$$

and define recursively the sets T_n^r for $r < n$ by setting $T_n^1 := U_n^1$ and for $r > 1$,

$$T_n^r := \left\{ (I, J, K) \in U_n^r \mid \forall p < r, (F, G, H) \in T_r^p : \sum_{f \in F} i_f + \sum_{g \in G} j_g \leqslant \sum_{h \in H} k_h + \frac{p(p-1)}{2} \right\}.$$

Horn [Hor62] conjectured that inequalities (IJK) for $(I, J, K) \in T_n^r$ are both necessary and sufficient, which was later proved to be true, see [Kly98; KT01].

Theorem 3.1.4 (Horn problem 1, [Kly98; KT01])**.** *There exists three Hermitian matrices (A, B, C) such that $A + B = C$ with respective eigenvalues α, β and γ such that (3.1.1) holds if and only if, for every $r < n$, the inequality (IJK) holds for every triple (I, J, K) in T_n^r.*

3.1.2 Enumerative geometry formulation

Geometric parametrization of inequalities

The goal of this section is to reinterpret the triples (I, J, K) appearing in the formulation of Theorem 3.1.4 from a geometric perspective. Inequalities (IJK) involve sums over subsets of indices in $[n]$, a type of sum which was studied by Hersch and Zwahlen [HZ62].

Recall that a *flag* $\mathcal{F}$ is a collection of nested subspaces

$$\{0\} = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n$$

such that $\dim(F_k) = k$ for all $0 \leqslant k \leqslant n$. Given a Hermitian matrix A with an orthonormal eigenbasis $x_1, \dots, x_n$, we denote by $\mathcal{F}(A)$ the flag defined by

$$F_k := \mathrm{Vect}(x_1, \dots, x_k), \text{ for } 1 \leqslant k \leqslant n.$$

This flag is called the *eigenflag* of the matrix A. Let us define the *Rayleigh trace* introduced by Fulton [Ful00].

Definition 3.1.5 (Rayleigh Trace)**.** Let $L \in \mathbb{G}r(k,n)$. Define $A_L : L \to L$ by

$$A_L x := P_L(Ax),$$

where P_L is the orthonormal projection from $\mathbb{C}^n$ onto L. The *Rayleigh trace* $R_A(L)$ is defined as

$$R_A(L) := \mathrm{Tr}[A_L].$$

Hersch and Zwahlen provided an expression of partial sums of eigenvalues using the Rayleigh trace.

Lemma 3.1.6 (Partial sum of eigenvalues, [HZ62])**.** *Let $\mathcal{F}(A)$ be an eigenflag of A. Then, for every $I = (i_1, \dots, i_k) \in \mathcal{P}_n^k$,*

$$\sum_{i \in I} \alpha_i = \min_{L \in \mathbb{G}r(k,n)} \left\{ \mathrm{Tr}[A_L] \mid \forall\ 1 \leqslant j \leqslant k, \dim(L \cap F_{i_j}) \geqslant j \right\}.$$

Therefore, partial sums of eigenvalues as the ones appearing in (IJK) inequalities from Horn's conjecture are related to the following subsets of the Grassmannians, known as *Schubert varieties.*

Definition 3.1.7 (Schubert variety)**.** Let $\mathcal{F}$ be a flag and let $I = (i_1, \dots, i_k) \in \mathcal{P}_n^k$ be distinct indices. The set

$$\Omega_I(\mathcal{F}) := \{L \in \mathbb{G}r(k,n) \mid \forall\ 1 \leqslant j \leqslant k, \dim(L \cap F_{i_j}) \geqslant j\}$$

is called the *Schubert variety* associated to the flag $\mathcal{F}$ and to the index tuple I.

Using this geometric formalism, one can rephrase Lemma 3.1.6 as

$$\sum_{i \in I} \alpha_i = \min_{L \in \Omega_I(\mathcal{F}(A))} \mathrm{Tr}[A_L].$$

This geometric interpretation leads to necessary conditions on eigenvalues due to [HR95; Tot94], based on the non-emptiness of intersections of Schubert varieties.

Let $I = (i_1 \dots i_k) \in \mathcal{P}_n^k$, and define its *complementary subset* $I' := (i'_1, \dots i'_k) \in \mathcal{P}_n^k$ by

$$i'_j = n + 1 - i_{k+1-j}, \text{ for } 1 \leqslant j \leqslant k.$$

Similarily, for a flag $\mathcal{F} = (F_0 \subset \cdots \subset F_n)$, define the *dual flag* $\mathcal{F}' = (F'_0 \subset \cdots \subset F'_n)$ as

$$F'_i := \mathrm{Vect}(v_{n+1-i}, \dots, v_n), \text{ for } 1 \leqslant i \leqslant n.$$

This construction is motivated by the observation that if $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ are the eigenvalues of A, then $-\alpha_n \geqslant \cdots \geqslant -\alpha_1$ are the eigenvalues of $-A$. Setting $\alpha'_i = -\alpha_i$, we have for any $I \in \mathcal{P}_n^k$:

$$\sum_{i \in I} \alpha_i = -\sum_{i \in I'} \alpha'_i.$$

Definition 3.1.8. Let $r < n$ and let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be flags. Define the set

$$S_n^r(\mathcal{F}, \mathcal{G}, \mathcal{H}) \coloneqq \{(I, J, K) \in U_n^r \mid \Omega_{I'}(\mathcal{F}) \cap \Omega_{J'}(\mathcal{G}) \cap \Omega_K(\mathcal{H}) \neq \emptyset\} \ . \tag{3.1.4}$$

Using Lemma 3.1.6 together with the equality $C - A - B = 0$, one derives the following necessary conditions.

Theorem 3.1.9 (Schubert inequalities, [Ful00])**.** *Let A, B and $C = A + B$ be Hermitian matrices with eigenvalues α, β and γ respectively. Then, for every $r < n$ and $(I, J, K) \in S_n^r(\mathcal{F}(A)', \mathcal{F}(B)', \mathcal{F}(C))$, the inequality* (IJK) *holds.*

Schubert calculus

Theorem 3.1.9 provides necessary inequalities that the triples (α, β, γ) must satisfy in order to occur as eigenvalues of Hermitian matrices A, B and $A + B$ respectively. The goal of this section is to give a criterion to determine whether a triple of indexes (I, J, K) belongs in the set

$$S_n^r(\mathcal{F}(A)', \mathcal{F}(B)', \mathcal{F}(C)).$$

Our exposition follows the approach of [Ful00] and Part 3 of [Ful97].

To translate the intersection condition for Schubert varieties, we need to introduce tools from the geometry of the Grassmannians. The Grassmannian $\mathbb{G}r(k, n)$ can be endowed with the structure of a compact, complex manifold of dimension $k(n - k)$ obtained from the quotient map

$$\begin{aligned} \pi : V_k^0(\mathbb{C}^n) &\longrightarrow \mathbb{G}r(k, n) \\ (u_1 \dots, u_k) &\longmapsto \mathrm{Vect}(u_1, \dots u_k) \end{aligned}$$

where $V_k^0(\mathbb{C}^n)$ denotes the set of k-tuples of orthonormal vectors in $\mathbb{C}^n$. Furthermore, the Plücker embedding

$$\mathbb{G}r(k, n) \hookrightarrow \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right)$$

realises the space $\mathbb{G}r(k, n)$ as an irreducible, non-singular projective variety. For $L \in \mathbb{G}r(k, n)$ and a flag $\mathcal{F}$, define the index tuple

$$I(L, \mathcal{F}) \coloneqq (i_1 < \cdots < i_k)$$

where for each $1 \leqslant j \leqslant k$,

$$i_j \coloneqq \min\{r \mid \dim(L \cap F_r) \geqslant j\} \ .$$

Definition 3.1.10 (Schubert cell)**.** Let $I = (i_1 < \cdots < i_k) \in \mathcal{P}_n^k$ and let $\mathcal{F}$ be a flag. The *Schubert cell* associated to I and $\mathcal{F}$ is the set

$$\begin{aligned} \Omega_I^\circ(\mathcal{F}) &\coloneqq \{L \in \mathbb{G}r(k, n) \mid I(L, \mathcal{F}) = I\} \\ &= \{L \in \mathbb{G}r(k, n) \mid \dim(L \cap F_k) = j, \text{ for } i_j \leqslant k \leqslant i_{j+1} - 1, 0 \leqslant j \leqslant k\}, \end{aligned}$$

where the condition for $j = 0$ being $L \cap F_{n-k+1-i_1} = 0$, see [Ful97].

The Schubert cell of a given index tuple I corresponds to all subspaces where the dimension of the intersection increases by intersecting with subspaces of $\mathcal{F}$ with indexes in I. Such cells partition the Grassmannian for any fixed flag $\mathcal{F}$:

$$\mathbb{G}r(k,n) = \bigsqcup_{I \in \mathcal{P}_n^k} \Omega_I^\circ(\mathcal{F}) \ .$$

We now relate the Schubert cells with the Schubert varieties of Definition 3.1.7.

Lemma 3.1.11 (Schubert varieties are closed Schubert cells, [Ful97])**.** *Let $I \in \mathcal{P}_n^k$ and let $\mathcal{F}$ be a flag. Then, the following equalities hold:*

$$\Omega_I(\mathcal{F}) = \overline{\Omega_I^\circ(\mathcal{F})} = \bigsqcup_J \Omega_J^\circ(\mathcal{F}),$$

where the closure is taken with respect to the Zariski topology and where the sum is on index tuples $J \in \mathcal{P}_n^k$ such that for every $1 \leqslant \ell \leqslant k$, $j_\ell \leqslant i_\ell$.

From Lemma 3.1.11, Schubert varieties are irreducible, closed subvarieties of the Grassmannian. The dimension of $\Omega_I(\mathcal{F})$ is given by

$$d(I) = \sum_{j=1}^{k} (i_j - j).$$

To each index tuple $I = (i_1 < \cdots < i_k) \in \mathcal{P}_n^k$, we associate a partition

$$\lambda(I) = (\lambda_1 \geqslant \cdots \geqslant \lambda_k)$$

defined by

$$\lambda(I)_j \coloneqq n - k + j - i_j, \text{ for } 1 \leqslant j \leqslant k.$$

We also define the size of a partition λ as

$$|\lambda| = \sum_{i=1}^{k} \lambda_i.$$

Let us recall the following results that can be found in [Ful97] and [Ful98b]. Set $X = \mathbb{G}r(k,n)$ and let us consider the singular homology groups $H_i(X)$ and cohomology groups $H^i(X)$ for $i \geqslant 0$. The cohomology $H^*(X) = \bigoplus_{i \geqslant 0} H^i(X)$ has a graded ring structure $H^\ell(X) \otimes H^k(X) \to H^{\ell+k}(X)$ given by the cup product.

Each Schubert variety $\Omega_I(\mathcal{F})$ gives a cohomology class $[\Omega_I(\mathcal{F})] \in H^{2|\lambda(I)|}(X)$ which does not depend on the choice of the flag. We denote it by $\sigma_I \coloneqq \Omega_I(\mathcal{F})$ and for a partition $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$, we write σ_λ to mean $\sigma_{I(\lambda)}$. The ring $H^*(X)$ is generated by the classes

$$\{\sigma_\lambda, n - k \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}.$$

For three partitions λ, μ and ν, let us denote by $c_{\lambda,\mu}^\nu \in \mathbb{N}$ the coefficient of σ_ν in the product $\sigma_\lambda \cdot \sigma_\mu$:

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda,\mu}^\nu \sigma_\nu \ , \tag{3.1.5}$$

where the sum is over partitions ν such that $|\lambda| + |\mu| = |\nu|$. The coefficients $c_{\lambda,\mu}^\nu$ in (3.1.5) are called the *Littlewood-Richardson coefficients*. The reason to consider the structure constants of the cohomology ring of the Grassmannian is that they provide an algebraic translation of intersections thanks to Kleiman's transversality theorem [Kle74]. This yields an algebraic criterion for when intersections of Schubert varieties are nonempty.

Theorem 3.1.12 (Characterization of nonempty intersection, [Ful98b; Kle74])**.** *The following are equivalent*

(i) The Littlewood–Richardson coefficient satisfies $c_{\lambda(I),\lambda(J)}^{\lambda(K)} > 0$

(ii) $(I, J, K) \in S_n^k$.

Using the characterization of Theorem 3.1.12 with Theorem 3.1.9 leads to the following result.

Corollary 3.1.13 (Littlewood-Richardson characterization of inequalities)**.** *Let* A, B *and* $C = A + B$ *be Hermitian matrices. Then, for every* $k < n$ *and triples* $(I, J, K) \in \mathcal{P}_n^k$*, such that* $c_{\lambda(I),\lambda(J)}^{\lambda(K)} > 0$*, the inequality* (IJK) *holds.*

Corollary 3.1.13 provides only a necessary condition on the eigenvalues. The work of Totaro [Tot94] and Klyachko [Kly98] give the converse and proves that these conditions of non-vanishing of Littlewood-Richardson coefficients are also sufficient. Their proof relies on the theory of stability of vector bundles and geometric invariant theory. We refer to [Ful00] and [Ful98a] for details on the work of Totaro and Klyachko.

Theorem 3.1.14 (Eigenvalues of a sum, [Ful00])**.** *Let* α, β *and* γ *be weakly decreasing sequences in* $\mathbb{R}^n$ *such that* $|\gamma| = |\alpha| + |\beta|$*. The following are equivalent*

(i) The inequality (IJK) *holds for all triples* $(I, J, K) \in \mathcal{P}_n^r$*, with* $r < n$ *such that* $c_{\lambda(I),\lambda(J)}^{\lambda(K)} > 0$.

(ii) There exist Hermitian matrices A*,* B*, and* C *with eigenvalues* α*,* β*, and* γ *respectively, such that* $A + B = C$.

This result raises a natural question: how do the inequalities (IJK) indexed by triples in S_n^r as in Theorem 3.1.14 compare to the recursive description of inequalities in T_n^r from Theorem 3.1.4 ? The answer is that the two sets coincide.

Theorem 3.1.15 ([KTW04])**.** *For every* $r < n$*, one has*

$$T_n^r = S_n^r.$$

While the full set of inequalities indexed by S_n^r, or equivalently T_n^r, completely characterizes the possible eigenvalues of Hermitian sums, some of these inequalities may be redundant. Knutson, Tao and Woodward [KTW04] identified a minimal subset of inequalities which imply the (IJK) inequalities for (I, J, K) in $T_n^r = S_n^r$ for every $r < n$. This minimal subset consists of inequalities parametrized by (I, J, K) in

$$R_n^r := \left\{(I, J, K) \in U_n^r \mid c_{\lambda(I),\lambda(J)}^{\lambda(K)} = 1\right\} \text{ for } r < n.$$

3.1.3 Littlewood-Richardson coefficients

The reduced subset of inequalities parametrized by R_n^r for $r < n$ was derived using a combinatorial representation of the Littlewood-Richardson coefficients. This model, introduced by Knutson and Tao [KT99] is called the *honeycomb model*. It was used to prove the saturation conjecture which states that for any triple of partitions λ, μ and ν,

$$c_{\lambda,\mu}^{\nu} \neq 0 \text{ if and only if } c_{N\lambda,N\mu}^{N\nu} \neq 0 \text{ for some } N \geqslant 1. \tag{3.1.6}$$

The proof of the saturation conjecture by Knutson and Tao [KT99], together with the work of Klyachko [Kly98] archieved to solve the Horn conjecture. An alternative but equivalent combinatorial model called *hives* was introduced in the same article. This description was used by Buch [Buc00] to give a proof of (3.1.6) without using honeycombs. The goal of this section is to present the various descriptions of the Littlewood-Richardson coefficients.

Combinatorial definition

The original formula given for the Littlewood-Richardson coefficients came from the work of Littlewood and Richardson [LR34], see also [Ful97] and [Mac79] from which this section borrows from. It is described in terms of Young tableaux, which we now introduce. Recall that $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$ is a sequence of weakly decreasing, non-negative integers.

Definition 3.1.16 (Ferrers diagram)**.** The *Ferrers diagram* of λ is the diagram consisting of n rows, where the i-th row for $1 \leqslant i \leqslant n$ contains λ_i boxes with rows aligned to the left.

The Ferrers diagram of the partition $\lambda = (6, 4, 4, 2)$ is shown in Figure 3.1. For two partitions λ and μ such that $\lambda_i \geqslant \mu_i$ for each i, one can define the *skew shape* $\lambda \setminus \mu$ by removing the boxes of the Ferrers diagram of μ from that of λ, see Figure 3.2 for an example.

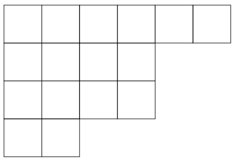

Figure 3.1: The Ferrers diagram of the partition $(6, 4, 4, 2)$.

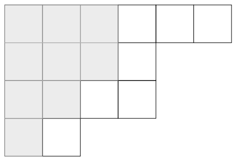

Figure 3.2: The Ferrers diagram of the skew shape $(6, 4, 4, 2) \setminus (3, 3, 2, 1)$. Gray boxes represent the Ferrers diagram of $\mu = (3, 3, 2, 1)$.

The Littlewood-Richardson coefficients are described in terms of fillings of skew-shapes, called semi-standard *Young tableaux*.

Definition 3.1.17 (Semi-standard Young tableau)**.** A *semi-standard Young tableau* of shape λ and content $\nu = (\nu_1 \geqslant \cdots \geqslant \nu_n)$ is a filling of the Ferrers diagram of λ with $|\nu|$ integers such that

- each integer $k \in [n]$ appears ν_k times,
- entries are weakly increasing in each row from left to right,
- entries are strictly increasing in each column from top to bottom.

We introduce an additional condition, called the *Yamanouchi condition*. It comes from ordering the boxes of a diagram from the top row to the bottom and from right to left within each row.

Definition 3.1.18 (Yamanouchi tableau)**.** Let T be a semi-standard Young tableau. We say that T satisfies the Yamanouchi condition, or that T is Yamanouchi if for any $p \geqslant 1$ and any $k \geqslant 1$, the number of times the integer k occurs in the first p boxes of T is greater or equal to the number of occurrences of $k + 1$.

Figure 3.1.3 shows two examples of Yamanouchi tableaux associated to the skew shape of Figure 3.2.
The Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is given by the number of Yamanouchi tableaux of skew shape $\nu \setminus \lambda$ having content μ.

Theorem 3.1.19 (Tableau expression of Littlewood-Richardson coefficients, [LR34])**.** *Let λ, μ and ν be partitions such that $|\nu| = |\lambda| + |\mu|$. Then, $c^{\nu}_{\lambda,\mu}$ is the number of Yamanouchi tableaux of skew shape $\nu \setminus \lambda$ having content μ.*

Figure 3.3: The two Yamanouchi tableaux of shape $(6,4,4,2) \setminus (3,3,2,1)$ and content $(4,2,1)$.

The hive model

In order to solve the saturation problem (3.1.6), Knutson and Tao [KT99] introduced a combinatorial model called *honeycombs* which give another expression of the Littlewood-Richardson coefficients. Honeycombs have an equivalent description in terms of *hives* that we now define.

Consider an equilateral triangle of length $n+1$ on which we draw the triangular lattice having edges of unit length. The faces of this lattice are equilateral triangles of unit length. We denote by T_n the set of vertices of this lattice. We call a *lozenge* or a *rhombus* a subset of four vertices of T_n that are vertices of two adjacent triangular faces. A lozenge has two acute vertices and two obtuse vertices. The central notion is the one of *rhombus concave function*.

Definition 3.1.20 (Rhombus concave function, hive)**.** A function $f : T_n \to \mathbb{R}$ is said to be *rhombus concave* if for every lozenge with acute vertices v_1 and v_2 and obtuse vertices v_3 and v_4, one has

$$f(v_1) + f(v_2) \leqslant f(v_3) + f(v_4).$$

The data of a rhombus concave function on T_n is called a *hive*, see Figure 3.4 for an example. The *boundary values* of a hive are given by reading successive differences on values of the function f on consecutive vertices, read in the directions depicted in Figure 3.4 for each boundary of T_n.

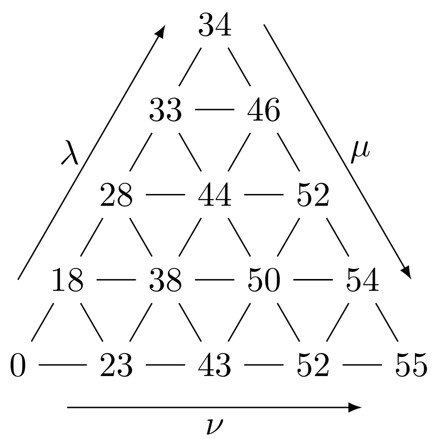

Figure 3.4: A hive on T_5 with boundary conditions $\lambda = (18, 10, 5, 1)$, $\mu = (12, 6, 2, 1)$ and $\nu = (23, 20, 9, 3)$.

Knutson and Tao [KT99] gave another combinatorial rule to compute Littlewood-Richardson coefficients by counting integer-valued rhombus concave functions with fixed boundary conditions.

Theorem 3.1.21 (Hive formula, [KT99; KT01])**.** *Let λ, μ, and ν be integer partitions. Denote by $T_{\mathbb{N}}(\lambda, \mu, \nu)$ the set of integral hives, that is, the set of integer-valued functions $f : T_n \to \mathbb{N}$ which are rhombus concave and whose boundary conditions are given by (λ, μ, ν). Then,*

$$c^{\nu}_{\lambda,\mu} = \#T_{\mathbb{N}}(\lambda, \mu, \nu) \ .$$

Knutson-Tao puzzles

We give another combinatorial rule to compute Littlewood-Richardson coefficients known as the *puzzle rule*. This rule is close to the hive description of the previous section. The puzzle rule was introduced in [KTW04] where it was shown to be equivalent to their honeycomb description and was used by Buch [Buc00] to provide an alternative version of the proof of the saturation conjecture adapted from [KT99]. For another approach of the puzzle rule that does not rely on honeycombs, we refer to [KT03].

Definition 3.1.22 (One-step puzzles)**.** A *one-step puzzle*, or just a puzzle is a tiling of T_N for some $N \geqslant 1$ using the set of edge-labeled tiles shown in Figure 3.5, such that adjacent pieces share the same labels on their common edges. The boundary values of a puzzle are the 02 strings obtained by reading the boundaries as in Figure 3.4.

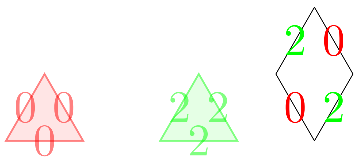

Figure 3.5: Pieces of one-step puzzles. Pieces can be rotated but not reflected.

The boundary labels of one-step puzzles are given by sequences of 0 and 2's. To a partition $\lambda = \lambda_1 \geqslant \cdots \geqslant \lambda_n$ such that $\lambda_1 \leqslant N - n$, for some $N \geqslant n$, we associate a sequence of 0's and 2's of size N, called a 02 *string* by reading the Ferrers diagram of λ from bottom to top and left to right inside a $n \times (N - n)$ rectangle. Horizontal and vertical steps correspond to labels 2 and 0 respectively, see Figure 3.6 for an example.

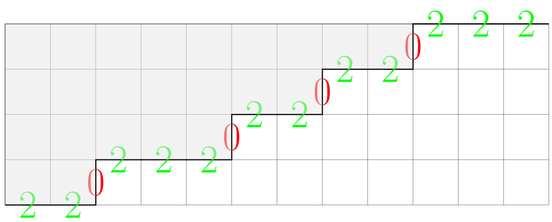

Figure 3.6: The partition $(9, 7, 5, 2)$ gives the string 2202220220220222 for $N = 16$.

Theorem 3.1.23 (Puzzle rule, [KTW04; KT03])**.** *Let $n \leqslant N$ and consider partitions λ, μ and ν such that their Ferrers diagrams are contained in a $n \times (N - n)$ rectangle. Then, the*

Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is equal to the number of puzzles of size N having boundaries given by the 02 *strings corresponding to λ, μ and ν.*

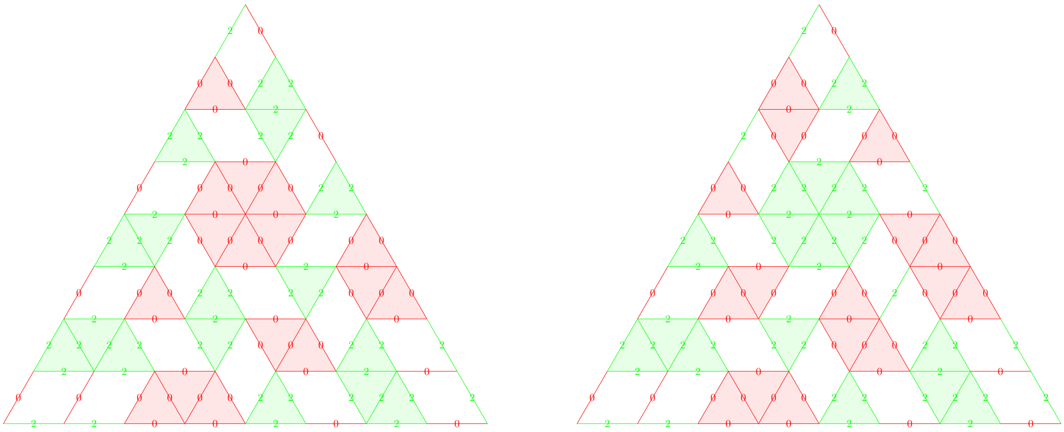

Figure 3.7: The two puzzles corresponding to $c^{\nu}_{\lambda,\mu} = 2$ for $\lambda = (0,1,2,3)$, $\mu = (0,1,2,2)$ and $\nu = (2,2,3,4)$ in $H^*(\mathbb{G}r(4,8))$. Pictures done with the module *Knutson-Tao puzzles* of Sage [The20].

Theorem 3.1.23 gives a combinatorial rule to compute intersections of Schubert varieties in the cohomology ring $H^*(\mathbb{G}r(n,N))$ since this ring has a basis consisting of classes $(\sigma_\lambda)_\lambda$ parametrized by partitions λ such that $\lambda_1 \leqslant N - n$, see Figure 3.7.
In [KT03], Knutson and Tao extend the set of pieces by considering the reflection of the rhombus in Figure 3.5. The corresponding puzzles are showed to compute the structure constants for the equivariant cohomology ring of the Grassmannians. Structure constants for other cohomology rings have later been expressed in terms of analogous puzzle rules, see Section 3.2.4.

3.1.4 Probabilistic version

In this section, we present a random matrix problem inspired by the deterministic setting of the additive Horn problem. This exposition follows the work of Coquereaux, McSwiggen and Zuber [CZ18; CMZ19; CMZ20].

Additive convolution of orbits

Let $\alpha = \alpha_1 \geqslant \cdots \geqslant \alpha_n$ and $\beta = \beta_1 \geqslant \cdots \geqslant \beta_n$ be real-valued n tuples. The set of Hermitian matrices with eigenvalues α is the *adjoint orbit* $\mathcal{O}^H(\alpha)$ of the diagonal matrix $\mathrm{Diag}(\alpha_1, \ldots, \alpha_n)$ under the adjoint action of the unitary group $\mathrm{U}(n)$:

$$\mathcal{O}^H(\alpha) := \{U \,\mathrm{Diag}(\alpha_1, \ldots, \alpha) U^*, U \in \mathrm{U}(n)\}.$$

Consider a random matrix $U \in \mathrm{U}(n)$ which is Haar distributed. The continuous map

$$\begin{aligned} f_\alpha : \mathrm{U}(n) &\to \mathcal{O}^H(\alpha) \\ U &\mapsto U \,\mathrm{Diag}(\alpha_1, \ldots, \alpha) U^* \end{aligned}$$

yields the pushforward measure m^H_α on the orbit $\mathcal{O}^H(\alpha)$ corresponding to the uniform measure on the orbit. This measure is called the *orbital measure*. Consider random matrices (A, B) independently distributed on respective orbits $\mathcal{O}^H_\alpha \times \mathcal{O}^H_\beta$, that is, having joint distribution $m^H_\alpha \otimes m^H_\beta$. Their sum $C = A + B$ yields a probability measure on the space of Hermitian matrices H_n whose eigenvalues $\gamma = \gamma_1 \geqslant \cdots \geqslant \gamma_n$ have a probability density function. The probabilistic version of the additive Horn problem asks:

What is the probability density of the eigenvalues $\gamma \in \mathbb{R}^n$ of $C = A + B$?

From Section 3.1.1, we know that this probability density is supported on a convex polytope in $\mathbb{R}^{n-1}$ whose faces are parametrized by the (IJK) inequalities.

Integrable formula for the probabilty density

A first step towards answering the previous question is to use Fourier transform in order to derive the probability distribution of the random matrix $C = A + B$ and then the probability density for its eigenvalues. This corresponds to computing the convolution of orbital measures, see [DRW93; Kir04] for more background on orbital measures.
The *Fourier transform*, or characteristic function of a random Hermitian matrix $A \in H_n$ with distribution m is given by

$$\hat{m}(X) := \mathbb{E}[\exp(i \operatorname{Tr}[XA])].$$

Let us denote by $m_* := m^H_\alpha * m^H_\beta$ the law of $C = A + B$ for $(A, B) \sim m^H_\alpha \otimes m^H_\beta$, that is, the law of the sum of two uniformly distributed matrices on orbits $\mathcal{O}^H_\alpha$ and $\mathcal{O}^H_\beta$. Then,

$$\hat{m}_*(X) = \hat{m}^H_\alpha(X) \cdot \hat{m}^H_\beta(X).$$

The Fourier transform of orbital measures was explicitly computed by Harish-Chandra [Har57] and Itzykson, Zuber [IZ80]. If D_α denotes the diagonal matrix $\operatorname{Diag}(\alpha_1, \ldots, \alpha_n)$,

$$\hat{m}_\alpha(X) = \int_{\mathrm{U}(n)} \exp(i \operatorname{Tr}[U D_\alpha U^* X]) \mathrm{d}U$$

By the left and right invariance of the Haar measure, the expression $m_\alpha(X)$ only depends on the spectra α and $x = (x_1, \ldots, x_n)$ of D_α and X respectively. We denote this function by $H(\alpha, x)$.

$$\hat{m}_\alpha(X) = \hat{m}_\alpha(D_x) = \int_{\mathrm{U}(n)} \exp(i \operatorname{Tr}[U D_\alpha U^* D_x]) \mathrm{d}U := H(\alpha, x).$$

Lemma 3.1.24 (Harish-Chandra-Itzykson-Zuber formula, [Har57; IZ80])**.** *One has*

$$H(\alpha, x) = \frac{\operatorname{sf}(n-1)}{i^{n(n-1)/2}} \cdot \frac{\det(\mathrm{e}^{i x_k \alpha_j})_{1 \leqslant j,k \leqslant n}}{\Delta(x)\Delta(\alpha)} \;, \tag{3.1.7}$$

where $\Delta(x) = \prod_{1 \leqslant j < k \leqslant n}(x_j - x_k)$ is the Vandermonde determinant of $(x_1, \ldots, x_n)$ and where $\operatorname{sf}(n) = \prod_{k=1}^n k!$.

Using Lemma 3.1.24 and applying the inverse Fourier transform yields the probability distribution for eigenvalues γ of $A + B$, see [Zub18].

Proposition 3.1.25 (Probability density for sum of adjoint orbits, [Zub18])**.** *Let $\gamma = (\gamma_1 \geqslant \cdots \geqslant \gamma_n)$ be the eigenvalues of $C = A + B$ where A and B are independent and uniformly distributed over $\mathcal{O}^H_\alpha$ and $\mathcal{O}^H_\beta$. Then,*

$$\mathrm{d}\mathbb{P}[\gamma|\alpha, \beta] = c_n \frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)} \int_{\mathbb{R}^n} \frac{1}{\Delta(x)} \det(\mathrm{e}^{i x_j \alpha_k}) \det(\mathrm{e}^{i x_j \beta_k}) \det(\mathrm{e}^{-i x_j \gamma_k}) \mathrm{d}x, \tag{3.1.8}$$

where c_n is a constant depending only on n.

The expression of the probability density in Proposition 3.1.25 is not explicitely positive as the determinants appearing in the integral are complex valued. We will follow the works [Zub18; CMZ20] and [CZ18] to give an explicit expression of (3.1.8) in terms of volumes. Computations and numerical simulations of the probability density for $n \leqslant 5$, can be found in [Zub18].

Volume formula

Recall that the support of $\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta]$ is an $n-1$ dimensional polytope supported on

$$\sum_{k=1}^{n}(\gamma_k - \alpha_k - \beta_k) = 0.$$

Let us write the density as done in [CZ18],

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{\mathrm{sf}(n-1)}{n!}\frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}J(\alpha,\beta,\gamma)\mathbb{1}_{\sum_{k=1}^{n}(\gamma_k-\alpha_k-\beta_k)=0}, \tag{3.1.9}$$

where the function J in (3.1.9) is called the *volume function*. We refer to [CZ18; CMZ20] for explicit expressions as integrals. The goal of the rest of this section is to give an explicit description of this volume function. Let $\mathbf{H}_{\alpha\beta}$ be the polytope of $\mathbb{R}^{n-1}$ for γ defined by the (IJK) inequalities of Theorem 3.1.14. The name volume function comes from the normalization with respect to the Lebesgue measure on $\mathbf{H}_{\alpha\beta}$,

$$\int_{\mathbf{H}_{\alpha\beta}} \frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}J(\alpha,\beta,\gamma)\mathrm{d}\gamma = \frac{1}{\mathrm{sf}(n-1)}.$$

The volume function vanishes for γ outside of $\mathbf{H}_{\alpha\beta}$. Inside $\mathbf{H}_{\alpha\beta}$ it is a non-negative piecewise polynomial which is homogeneous of degree $\frac{1}{2}(n-1)(n-2)$ in α,β and γ. It is also antisymmetric in its arguments. The link between the volume function J and Littlewood-Richardson coefficients lies in the asymptotic expansion, which is a particular case of limits of large dimensional representations studied in [GLS96] and [Hec82].

Proposition 3.1.26 (Volume function and Littlewood-Richardson, see [CZ18])**.** *Let α,β and γ be n-tuples of real numbers and partitions λ,μ and ν such that*

$$\frac{1}{N}\lambda \to \alpha,\ \ \frac{1}{N}\mu \to \beta \ \textit{and}\ \frac{1}{N}\nu \to \gamma$$

entrywise. Then, as $N \to \infty$

$$J(\alpha,\beta,\gamma) = \lim_{N\to\infty} N^{-\frac{(n-1)(n-2)}{2}} c_{N\lambda,N\mu}^{N\nu}$$

Let $H_{\alpha\beta}^{\gamma}$ be the polytope of hives with real boundary conditions α,β and γ, see Definition 3.1.20. From the hive model of Knutson and Tao [KT99], the Littlewood-Richardson coefficient $c_{\lambda,\mu}^{\nu}$ is the number of integral valued hives, that is, the number of integral points in $H_{\lambda,\mu}^{\nu}$. From the relation (3.1.26), the volume function corresponds to the limit of the number of integral points in the streched rational polytope $H_{N\lambda,N\mu}^{N\nu}$ of dimension $d = (n-1)(n-2)/2$ normalized by N^{-d}. This limits corresponds to the Lebesgue volume of the polytope. One derives the volume expression for the function J.

Theorem 3.1.27 (Volume expression, [CZ18]). *One has*

$$J(\alpha, \beta, \gamma) = \mathrm{Vol}(H_{\alpha,\beta}^{\gamma}), \tag{3.1.10}$$

so that

$$\mathrm{d}\mathbb{P}[\gamma|\alpha, \beta] = \frac{\mathrm{sf}(n-1)}{n!} \frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)} \mathrm{Vol}(H_{\alpha,\beta}^{\gamma}). \tag{3.1.11}$$

Remark 3.1.28 (Symplectic geometry approach)**.** An alternative derivation of the volume formula uses symplectic geometry and moment maps, as described in Knutson's survey [Knu00]. In particular, Theorem 4 in [KT01] expresses the probability density as the volume of a symplectic quotient. An open question raised by Knutson and Tao is to construct an explicit measure-preserving map from the symplectic quotient to the hive polytope.

3.2 Products of unitary matrices

This section presents an analog of the additive Horn problem from Section 3.1, known as the multiplicative or unitary Horn problem. This problem concerns products of unitary matrices. As in the Hermitian version, the goal is to characterize the possible eigenvalues of such products when the spectra of the individual matrices are prescribed. Section 3.2.1 presents this latter problem from a linear algebraic perspective. As the Hermitian Horn problem of Section 3.1, the unitary Horn problem involves combinatorial coefficients called *quantum Littlewood-Richardson coefficients*, which count certain rational maps. Section 3.2.2 explains how the multiplicative Horn problem relates to this counting problem. Quantum Littlewood-Richardson coefficients are related to the quantum cohomology of the Grassmannians, which is introduced in Section 3.2.3. As with the classical Littlewood-Richardson coefficients, a puzzle rule has been established to compute the quantum version. This puzzle rule is presented in Section 3.2.4. Finally, products of unitary matrices are related to the computation of volumes of moduli spaces of flat connections on Riemann surfaces. This geometric perspective is presented in Section 3.2.5.

3.2.1 Eigenvalues of products of unitary matrices

The multiplicative or unitary Horn problem asks the following question:

Given two unitary matrices, which eigenvalues can arise for their product ?

This question was answered by Agnihotri and Woodward [AW98] who gave inequalities that determine the possible eigenvalues of a product of unitary matrices. At the same time, Belkale [Bel01] addressed the same question by solving a problem of Katz [Kat96] on local systems over the Riemann sphere. Biswas [Bis98] had previously solved the multiplicative Horn problem in dimension $n = 2$.

Let $n \geqslant 1$ be an integer and let $A, B \in \mathrm{U}(n)$ be unitary matrices. Up to shifts by $\det(A)$ and $\det(B)$ which are complex numbers of modulus 1, one can assume that A and B have unit determinant. Therefore, in this section, we consider matrices in the special unitary group

$$\mathrm{SU}(n) \coloneqq \{U \in \mathcal{M}_n(\mathbb{C}) \mid U^*U = I_n,\ \det(U) = 1\}\,.$$

For $A \in \mathrm{SU}(n)$, its eigenvalues lie on the unit circle $\mathbb{S}^1$ and can be parametrized by angles

$$\alpha = (\alpha_1 \geqslant \cdots \geqslant \alpha_n) ,$$

where $\alpha_k \in [0,1]$ for $1 \leqslant k \leqslant n$ and such that $\sum_{1\leqslant k\leqslant n} \alpha_k \in \mathbb{N}$. Let us denote by

$$\mathcal{O}(\alpha) := \left\{ U \operatorname{Diag}(\mathrm{e}^{2i\pi\alpha_1}, \dots, \mathrm{e}^{2i\pi\alpha_n}) U^*, U \in \mathrm{U}(n) \right\}$$

the orbit of α, that is, the matrices in $\mathrm{SU}(n)$ having eigenvalues $\mathrm{e}^{2i\pi\alpha_1}, \dots, \mathrm{e}^{2i\pi\alpha_n}$. The results of Agnihotri, Woodward [AW98] and Belkale [Bel01], give necessary and sufficient inequalities relating eigenvalues of matrices (A, B, C) satisfying the relation $ABC = 1$. Let us introduce the corresponding matrix space for any number of factors $A_1, \dots, A_\ell$ with $\ell \geqslant 1$ and prescribed orbits $(\theta_1, \dots, \theta_\ell)$ where $\theta_k = (\theta_{k,1} \geqslant \cdots \geqslant \theta_{k,n})$:

$$\{(A_1, \dots, A_\ell) \in \mathcal{O}(\theta_1) \times \cdots \times \mathcal{O}(\theta_\ell) \mid A_1 \cdots A_\ell = I_n\} .$$

The previous set is stable by conjugation of each factor. Let us denote the quotient by

$$\mathcal{M}(\theta_1, \dots, \theta_\ell) := \{(A_1, \dots, A_\ell) \in \mathcal{O}(\theta_1) \times \cdots \times \mathcal{O}(\theta_\ell) \mid A_1 \cdots A_\ell = I_n\} /\mathrm{SU}(n) . \quad (3.2.1)$$

The multiplicative Horn problem therefore asks:

which tuples (α, β, γ) give an non-empty set $\mathcal{M}(\alpha, \beta, \gamma)$?

3.2.2 Representation of fundamental group and parabolic bundles

Representation theory aspect

Similar to the additive Horn problem, the description of possible tuples $(\theta_1, \dots, \theta_\ell)$ for which $\mathcal{M}(\theta_1, \dots, \theta_\ell)$ is not-empty relies on inequalities parametrized by integer-valued coefficients called *Gromov–Witten invariants* which count rational maps. Before introducing these coefficients in the next section, we provide motivation for why counting rational maps is relevant for the description of $\mathcal{M}(\theta_1, \dots, \theta_\ell)$.

Let $\mathbb{P}^1_{\mathbb{C}}$ be the Riemann sphere. For $\ell \geqslant 1$, the set $\mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, \dots, p_\ell\}$ is called the ℓ-holed sphere. The fundamental group of the ℓ holed sphere, based at a point $x \in \mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, \dots, p_\ell\}$, denoted by

$$\pi_1 := \pi_1 \left(\mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, \dots, p_\ell\}, x \right)$$

is the free group with ℓ generators $c_1, \dots, c_\ell$ corresponding to loops around each punctured point $p_1, \dots, p_\ell$ with the relation $c_1 \cdots c_\ell = 1$:

$$\pi_1 = \langle c_1, \dots, c_\ell \mid c_1 \cdots c_\ell = 1 \rangle.$$

A representation $\rho : \pi_1 \to \mathrm{SU}(n)$ of the fundamental group therefore is determined by the unitary matrices $\rho(c_1), \dots, \rho(c_\ell)$. The space $\mathcal{M}(\theta_1, \dots, \theta_\ell)$ can be viewed as the space of representations of the fundamental group of the ℓ-holed sphere with fixed conjugacy classes for the generators as described in [AW98; Bel01].

$$\mathcal{M}(\theta_1, \dots, \theta_\ell) \simeq \{\rho : \pi_1 \to \mathrm{SU}(n) \mid \rho(c_k) \in \mathcal{O}(\theta_k), 1 \leqslant k \leqslant \ell\} /\mathrm{SU}(n). \quad (3.2.2)$$

Parabolic bundles

By the previous section, the multiplicative Horn problem is equivalent to providing representations of the fundamental group of the $\ell = 3$ holed sphere. A geometric condition equivalent to the existence of these representations can be stated in terms of existence of *parabolic vector bundles* over $\mathbb{P}^1_{\mathbb{C}}$ that we introduce in this section. Our presentation follows [AW98; Bel01] and relies on the work of Mehta and Seshadri [MS80] from which the main result of this section is derived.

Definition 3.2.1 (Parabolic vector bundle)**.** Let C be a Riemann surface with ℓ marked points $p_1, \ldots, p_\ell$. A *parabolic vector bundle* over C consists of a rank n vector bundle V, together with:

- for each $1 \leqslant k \leqslant \ell$, a complete flag of the fiber V_{p_k} at p_k,

$$V_k : 0 = V_{k,0} \subset V_{k,1} \subset \cdots \subset V_{k,n} = V_{p_k} \ ,$$

- real-valued weights $a_{k,1} \geqslant a_{k,2} \geqslant \cdots \geqslant a_{k,n} \geqslant a_{k,1} - 1$.

The *parabolic degree* of V is defined as

$$\operatorname{pardeg}(V) \coloneqq \deg(V) + \sum_{k=1}^{\ell} \sum_{i=1}^{n} a_{k,i} \ ,$$

where the degree of a vector bundle is the one of the first Chern class $c_1 \in H^2(C, \mathbb{Z}) \simeq \mathbb{Z}$.

Subbundles of parabolic vector bundles can be endowed with a parabolic structure as follows. Let $W \subset V$ be a subbundle of rank r of a parabolic vector bundle V. For each marked point $p_k, 1 \leqslant k \leqslant \ell$, the complete flag W_k of the fiber W_{p_k} is given by

$$0 = W_{k,0} \subset W_{k,1} = (W_{p_k} \cap V_{k,1}) \subset \cdots \subset W_{k,n} = (W_{p_k} \cap V_{k,n}) = W_{p_k},$$

where subspaces with equal dimension are removed to yield a complete flag. For $1 \leqslant k \leqslant \ell, 1 \leqslant i \leqslant r$, the weights $(a'_{k,i})$ of W are defined as

$$a'_{k,i} = a_{k,j} \text{ where } j = \inf\{s \geqslant 1 : W_{k,i} = W_k \cap V_{k,s}\}.$$

The central notion related to the existence of representations is the one of stability.

Definition 3.2.2 (Stability)**.** A parabolic vector bundle V is called *stable*, respectively *semi-stable*, if for every subbundle W of V, one has $\operatorname{pardeg}(W) < \operatorname{pardeg}(V)$, respectively $\operatorname{pardeg}(W) \leqslant \operatorname{pardeg}(V)$.

According to the work of Mehta and Seshadri [MS80] and as reviewed in the appendix of [Bel01], the existence of (semi-)stable parabolic vector bundles is equivalent to the existence of (irreducible) representations of the fundamental group. We consider ℓ conjugacy classes in $\mathrm{SU}(n)$ with parameters

$$\theta_{k,i}, \ 1 \leqslant k \leqslant \ell, 1 \leqslant i \leqslant n.$$

Without loss of generality, one can assume that for each k,

$$\sum_{i=1}^{n} \theta_{k,i} = 0 \text{ and } \theta_{k,1} - \theta_{k,n} \leqslant 1.$$

Theorem 3.2.3 (Representations and stability, [MS80]). *There exists a representation (respectively an irreducible representation)*

$$\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, \dots, p_\ell\}, x) \to \mathrm{SU}(n)$$

such that $\rho(c_k) \in \mathcal{O}(\theta_k)$ for $1 \leqslant k \leqslant \ell$ if and only if there exists a semi-stable (respectively stable) parabolic bundle of parabolic degree zero with weights $\theta_{k,i}$.

We are now left to understand the semi-stable parabolic bundles over $\mathbb{P}^1_{\mathbb{C}}$ which is a condition on any subbundle over $\mathbb{P}^1_{\mathbb{C}}$. From [AW98; Bel01], subbundles of rank r and degree $-d$ are in correspondance with holomorphic maps

$$f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{G}r(r, n)$$

of degree d, where the degree a map f is defined as the integer given by homology class $[f] \in H^2(\mathbb{G}r(r, n), \mathbb{Z}) \simeq \mathbb{Z}$. This correspondance is obtained by pulling back the tautological bundle S over $\mathbb{G}r(r, n)$ defined as $S = \{(V, v), V \in \mathbb{G}r(r, n), v \in V\}$.

The pullback bundle f^*S has degree $-d$ and inherits a parabolic structure from the bundle $\mathbb{C}^n$ over $\mathbb{P}^1_{\mathbb{C}}$ having weights $\theta_{k,i}$. For each $1 \leqslant k \leqslant \ell$, the image point $f(p_k) \in \mathbb{G}r(r, n)$ lies in a Schubert variety $\Omega_{I_k}(\mathcal{F}_k)$ for some generic flag $\mathcal{F}_k$ and index tuple $I_k = (i_{k,1} < \cdots < i_{k,r})$. The weights of the parabolic subbundle f^*S are then

$$\theta_{k,i}, \ 1 \leqslant k \leqslant \ell, i \in I_k.$$

Therefore, the semi-stability condition

$$\mathrm{pardeg}(f^*S) \leqslant 0$$

can be written

$$\sum_{k=1}^{\ell} \sum_{i \in I_k} \theta_{k,i} \leqslant d.$$

We thus obtain inequalities parametrised by subsets $I_1, \dots, I_\ell$ of size r and integers $d \geqslant 0$ which are valid whenever there exists a rational map $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{G}r(r, n)$ of degree d which passes through the Schubert varieties associated with the subsets I_k. These inequalities determine the existence of semi-stable parabolic bundles and hence, of representations by Theorem 3.2.3.

3.2.3 Quantum Cohomology of the Grassmannians

Gromov–Witten invariants

In the light of the inequalities obtained at the end of Section 3.2.2, one must characterize the number of rational maps from the Riemann sphere to the Grassmannians passing though prescribed Schubert varieties. Such numbers are called *Gromov–Witten* invariants. We refer to [MS04] for a detailed construction of these invariants.

Definition 3.2.4 (Gromov–Witten invariants). Let $(I_1, \dots, I_\ell) \in (\mathcal{P}^r_n)^\ell$ be index subsets of $[n]$ of size $r \leqslant n$ and let $d \geqslant 0$ be an integer. Let $p_1, \dots, p_\ell$ be points in $\mathbb{P}^1_{\mathbb{C}}$ and $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ be flags. The *Gromov–Witten* invariant $\langle \sigma_{I_1}, \dots, \sigma_{I_\ell} \rangle_d$ is the number of holomorphic maps $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{G}r(r, n)$ of degree d such that for each $1 \leqslant k \leqslant \ell$, $f(p_k) \in \Omega_{I_k}(\mathcal{F}_k)$. This number is set to be zero if there is an infinite number of such maps.

We now state the main theorem that solves the multiplicative Horn problem established by Agnihotri, Woodward [AW98] and Belkale from [Bel01].

Theorem 3.2.5 (Characterisation of eigenvalues for products of unitary matrices, [AW98; Bel01])**.** *Let $n \geqslant 1$. Let $(\theta_1, \ldots, \theta_\ell)$ be such that for every $1 \leqslant k \leqslant \ell$,*

$$\theta_{k,1} \geqslant \cdots \geqslant \theta_{k,n}, \ \sum_{i=1}^{n} \theta_{k,i} = 0 \text{ and } \theta_{k,1} - \theta_{k,n} \leqslant 1.$$

Then, there exists matrices $A_1, \ldots, A_\ell \in \mathrm{SU}(n)$ such that for each $1 \leqslant k \leqslant \ell$, $A_k \in \mathcal{O}(\theta_k)$ satisfying $A_1 \cdots A_\ell = I_n$ if and only if the inequalities

$$\sum_{k=1}^{\ell} \sum_{i \in I_k} \theta_{k,i} \leqslant d.$$

hold for every $I_1, \ldots, I_\ell$ of size $r < n$ such that $\langle \sigma_{I_1}, \ldots, \sigma_{I_\ell} \rangle_d > 0$.

A shortened list of necessary and sufficient inequalities is given by replacing the condition $\langle \sigma_{I_1}, \ldots, \sigma_{I_\ell} \rangle_d > 0$ by $\langle \sigma_{I_1}, \ldots, \sigma_{I_\ell} \rangle_d = 1$, see [Bel01]. Theorem 3.2.5 gives a characterization for products with any number of factors ℓ while the multiplicative Horn problem deals with $\ell = 3$. In the rest of this section, we will consider the case $\ell = 3$.

Quantum cohomology ring

Recall that the cohomology ring of the Grassmannian $H^*(\mathbb{G}r(n,N))$ has a basis $(\sigma_\lambda)_\lambda$ given by cohomology classes associated to Schubert varieties parametrized by non-increasing sequences of integers $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$ with $\lambda_n \geqslant 0$ and $\lambda_1 \leqslant N - n$. Such a sequence λ is equivalent to an increasing subset $I(\lambda) = 1 \leqslant i_1 < \cdots < i_n \leqslant N$ with

$$i_j = N - n + j - \lambda_j, 1 \leqslant j \leqslant n \ .$$

For a subset $I(\lambda)$, denote by I' the subset $i'_1 < \cdots < i'_n$ defined by

$$i'_j = N + 1 - i_{n+1-j}, 1 \leqslant j \leqslant n$$

which is the subset $I(\lambda^\vee)$ where

$$\lambda^\vee_j = N - n - \lambda_{n+1-j}, 1 \leqslant j \leqslant n \ .$$

For three non-increasing sequences λ, μ, ν with respective associated subsets I, J, K and an integer $d \in \mathbb{N}$ called the degree of the coefficient, set

$$c^{\nu,d}_{\lambda,\mu} \coloneqq \langle \sigma_I, \sigma_J, \sigma_{K'} \rangle_d = \langle \sigma_\lambda, \sigma_\mu, \sigma_{\nu^\vee} \rangle_d \ ,$$

where for partitions λ, μ, ν,

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \langle \sigma_{I(\lambda)}, \sigma_{I(\mu)}, \sigma_{I(\nu)} \rangle_d \ .$$

The quantum cohomology ring of the Grassmannian is a deformation of the cohomology ring $H^*(\mathbb{G}r(n,N))$ introduced by Vafa [Vaf92], see also the work of Bertram [Ber97] and Chapter 11 of [MS04]. It is defined as

$$QH^*(\mathbb{G}r(n,N), \mathbb{Z}) \coloneqq H^*(\mathbb{G}r(n,N), \mathbb{Z}) \otimes \mathbb{Z}[q] \ ,$$

with basis $(\sigma_\lambda \otimes 1)_\lambda$ as a $\mathbb{Z}[q]$ module. For simplicity, we write $q^d\sigma_\lambda$ for $\sigma_\lambda \otimes q^d$. The ring structure is given by the quantum product

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu, d \geqslant 0} c_{\lambda,\mu}^{\nu,d} q^d \sigma_\nu$$

where the sum is over non-increasing sequences ν such that

$$|\lambda| + |\mu| = |\nu| + Nd \ .$$

We refer to [RT94; KM97] for the non-trivial fact that the above product yields an associative ring structure. The first term in the sum corresponds to classical Littlewood-Richardson coefficients since

$$c_{\lambda,\mu}^{\nu,0} = c_{\lambda,\mu}^{\nu} \ .$$

3.2.4 Puzzle rule

This section presents a combinatorial rule to compute quantum Littlewood-Richardson coefficients. We first show that Gromov–Witten invariants are particular instances of structure constants for the cohomology ring of the two-step flag variety. This result is based on Buch's span and kernel bijection [Buc03; BKT03]. Buch, Kresch, Purbhoo and Tamvakis [Buc+16] proved a conjecture of Knutson, showing that the structure constants of the cohomology ring of the two-step flag variety can be computed using a puzzle rule. In particular, this provides a puzzle rule for computing quantum cohomology coefficients as presented in Corollary 3.2.10.

Two-step flag variety

The two-step flag variety is the data of two subspaces, one included in the other, with prescribed dimensions.

Definition 3.2.6 (Two-step flag variety)**.** Let $N \geqslant 1$ be an integer. For $1 \leqslant k \leqslant n \leqslant N$, the *two-step flag variety* $F(k, n, N)$ is defined as

$$F(k, n, N) \coloneqq \{(A, B) \mid A \subset B \subset \mathbb{C}^N, \dim(A) = k, \dim(B) = n\}.$$

The two-step flag variety also admits a decomposition by Schubert varieties as in the case of the Grassmannians. Schubert varieties of the two-step flag variety $F(k, n, N)$ are parametrized by permutations $w \in S_N$.

Definition 3.2.7 (Schubert variety in $F(a_1, a_2, N)$, [BKT03])**.** Let $w \in S_N$ be a permutation such that $w(i) < w(i+1)$ for $i \notin \{a_1, a_2\}$ and let $\mathcal{F} : F_0 \subset \cdots \subset F_N$ be a flag in $\mathbb{C}^N$. The *Schubert variety* $X_w(\mathcal{F})$ of the two-step flag variety $F(a_1, a_2, N)$ is defined by

$$X_w(\mathcal{F}) \coloneqq \left\{ (A_1, A_2) \in F(a_1, a_2, N) \mid \begin{matrix} \dim(A_i \cap F_j) \geqslant \#\{p \leqslant a_i \mid w(p) > N - j\} \\ i \in \{1, 2\}, 1 \leqslant j \leqslant N \end{matrix} \right\}.$$

Schubert varieties $\Omega_\lambda(\mathcal{F}) = \Omega_{I(\lambda)}(\mathcal{F})$ of $\mathbb{G}r(n, N)$ can also be parametrized by permutations $w \in S_N$. To $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$, one associates the unique permutation $w = w_\lambda \in S_N$ satisfying $w(i) = \lambda_{n+1-i} + i$ for $i \leqslant n$ and $w(i) < w(i+1)$ for $k+1 \leqslant i \leqslant N$. Then,

$$\Omega_\lambda(\mathcal{F}) = \{L \in \mathbb{G}r(n, N) \mid \dim(L \cap F_j) \geqslant \#\{p \leqslant n \mid w(p) > N - j\}, 1 \leqslant j \leqslant N\} \, .$$

The main result of Buch [Buc03] is to associate to any rational map $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{G}r(n, N)$ of degree d an element in $F(n-d, n+d, N)$ given by two subspaces, called the kernel and the span of the map which allows for simpler computations in the quantum cohomology ring.

Definition 3.2.8 (Span and kernel of a curve, [Buc03; BKT03]). Let $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{G}r(n, N)$ be a rational map of degree $d \geqslant 0$. The *span* and *kernel* of f are defined as

$$\mathrm{Span}(f) \coloneqq \sum_{p \in \mathbb{P}^1_{\mathbb{C}}} f(p), \;\; \mathrm{Ker}(f) \coloneqq \bigcap_{p \in \mathbb{P}^1_{\mathbb{C}}} f(p).$$

The space $\mathrm{Span}(f)$ is the smallest subspace containing all the n dimensional subspaces $f(p)$ while $\mathrm{Ker}(f)$ is the largest subspace contained in every subspace $f(p)$. We give the main result of [BKT03]. To a subvariety $X \subset \mathbb{G}r(n, N)$, we associate

$$X^{(d)} \coloneqq \{(A, B) \in F(n-d, n+d, N) \mid A \subset X \subset B\} \;\; ,$$

which is a subvariety of $F(n-d, n+d, N)$. In particular, if $\Omega_\lambda(\mathcal{F})$ is a Schubert variety in $\mathbb{G}r(n, N)$, $\Omega_\lambda^{(d)}(\mathcal{F})$ is a Schubert variety in $F(n-d, n+d, N)$ associated to the permutation obtained from w_λ by sorting the values of $w(n-d+1), \dots, w(n+d)$ in the increasing order, see [BKT03]. The main result of [BKT03] is the following.

Theorem 3.2.9 (Quantum to classical bijection, [BKT03]). *Let λ, μ, ν be partitions such that $|\lambda| + |\mu| + |\nu| = n(N-n) + Nd$ and let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be flags of $\mathbb{C}^N$. The map*

$$f \mapsto (\mathrm{Ker}(f), \mathrm{Span}(f))$$

is a bijection between the set of holomorphic maps $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{G}r(n, N)$ of degree d such that $f(0) \in \Omega_\lambda(\mathcal{F}), f(1) \in \Omega_\mu(\mathcal{G})$ and $f(\infty) \in \Omega_\nu(\mathcal{F})$ with the set of points in the intersection $\Omega_\lambda^{(d)}(\mathcal{F}) \cap \Omega_\mu^{(d)}(\mathcal{G}) \cap \Omega_\nu^{(d)}(\mathcal{H})$ in $F(n-d, n+d, N)$.

From Theorem 3.2.9, Buch, Kresch and Tamvakis derive an expression of the quantum Littlewood-Richardson number as classical intersection numbers in the cohomology ring of the two-step flag variety.

Corollary 3.2.10 ([BKT03]). *If $\left[\Omega_\lambda^{(d)}\right]$ denotes the cohomology class of $\Omega_\lambda^{(d)}$ in the cohomology ring $H^*(F(n-d, n+d, N))$,*

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \int_{F(n-d,n+d,N)} [\Omega_\lambda^{(d)}] \cdot [\Omega_\mu^{(d)}] \cdot [\Omega_\nu^{(d)}].$$

In [BKT03], it was conjectured that a puzzle rule would give a combinatorial way of computing the intersection number of Corollary 3.2.10. This conjecture was proved by Buch, Kresch, Purbhoo and Tamvakis in the work [Buc+16] presented in the next section.

Puzzle rule for two-step variety

Thanks to Corollary 3.2.10, quantum Littlewood-Richardson coefficients are particular instances of structure constants for the cohomology ring $H^*(F(k, n, N))$ which has basis $([X_w])_w$ given by Schubert varieties of Definition 3.2.7. As presented in [Buc+16], it is more convenient to index Schubert varieties of $F(k, n, N)$ by words of length N in the alphabet $\{0, 1, 2\}$ called 012 *strings* having k labels 0, $n-k$ labels 1 and $N-n$ labels 2. A permutation $w \in S_N$ corresponds to the 012 string where 0's are in positions $w(1), \ldots, w(k)$, 1's are in positions $w(k+1), \ldots, w(n)$ and the remaining entries are 2's, as described in [BKT03]. For a 012 string u, we write X_u for the Schubert variety X_w of the associated permutation w.

In the case where the Schubert variety arises from a partition λ, that is, if

$$X_w = \Omega_\lambda^{(d)} \in F(n-d, n+d, N),$$

the 012 string $u = u(\lambda)$ such that $X_w = X_u$ can be obtained by replacing the first d occurrences of 2 and the last d occurrences of 0 in the 02 code of λ of Figure 3.6 by 1's, see Figure 3.8 for an example.

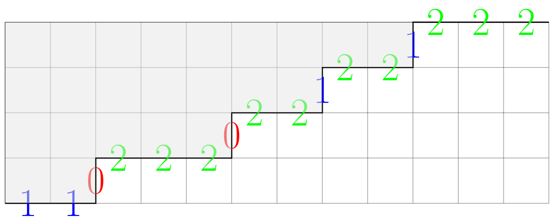

Figure 3.8: The partition $(9, 7, 5, 2)$ gives the string 1102220221221222 for $N = 16$.

The main result of [Buc+16] states that for three Schubert varieties X_u, X_v and X_w associated to 012 strings u, v and w, the intersection number

$$\int_{F(k,n,N)} [X_u] \cdot [X_v] \cdot [X_w]$$

can be computed by a puzzle rule thereby generalizing Theorem 3.1.23 on one-step puzzles of Definition 3.1.22.

Definition 3.2.11 (Two-step puzzle, [Buc+16])**.** A *two-step puzzle* is a tiling of T_N for some $N \geqslant 1$ by the set of edge labeled tiles of Figure 3.5 such that labels of adjacent pieces coincide.

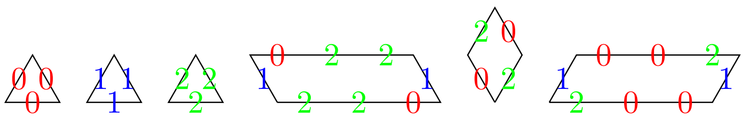

Figure 3.9: Tiles used in two-step puzzles. Tiles can be rotated. Pieces four and six may contain an arbitrary number of 2s and 0s, respectively.

Theorem 3.2.12 (Puzzle rule for two-step flag variety, [Buc+16])**.** *Let X_u, X_v and X_w be three Schubert varieties in $F(k, n, N)$. Then, the number*

$$\int_{F(k,n,N)} [X_u] \cdot [X_v] \cdot [X_w]$$

is given by the number of two-step puzzles for which u, v and w are the respective boundary labels of left, right and bottom sides in clockwise order.

As a corollary, the quantum Littlewood-Richardson coefficient $c_{\lambda,\mu}^{\nu,d}$ for $|\lambda|+|\mu|=|\nu|+Nd$ is equal to the number of two-step puzzles with boundary 012 strings associated to λ,μ and $\nu^\vee$, see an example in Figure 3.10.

$$c_{\lambda,\mu}^{\nu,d}=\langle\sigma_\lambda,\sigma_\mu,\sigma_{\nu^\vee}\rangle=\int_{F(n-d,n+d,N)}[X_\lambda^{(d)}]\cdot[X_\mu^{(d)}]\cdot[X_{\nu^\vee}^{(d)}].$$

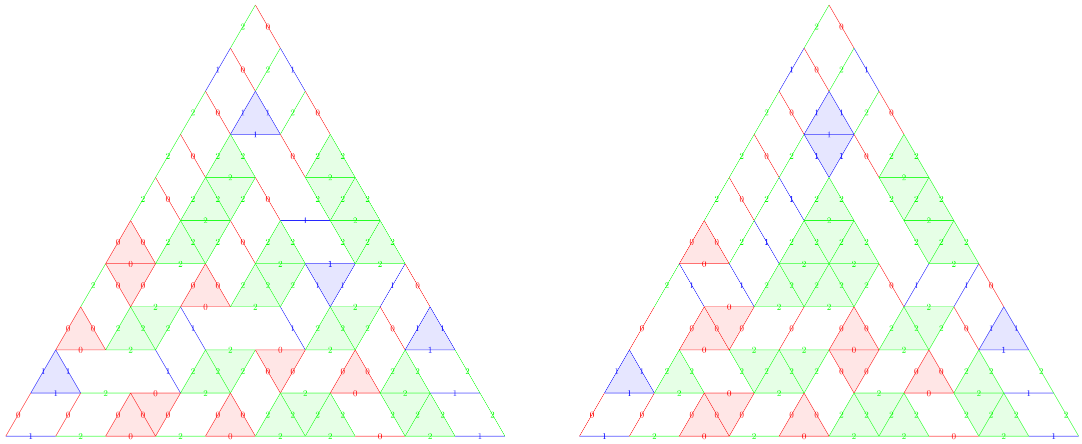

Figure 3.10: The two puzzles corresponding to $c_{\lambda,\mu}^{\nu,d}=2$ for $\lambda=(5,2,1,0)$, $\mu=(4,4,1,0)$ and $\nu=(4,2,1,0)$ in $QH^*(\mathbb{G}r(4,10))$ and $d=1$. Pictures done with the module *Knutson-Tao puzzles* of Sage [The20].

Finally, we mention an alternative positive formula for computing the structure constants of the cohomology ring of the two-step flag variety, due to Coskun [Cos09]. This approach uses a sequence of diagrams known as *Mondrian tableaux*, which record intersections of Schubert varieties.

3.2.5 Moduli space of flat connections

Principal bundles and connections

This section establishes the relation between the multiplicative Horn problem and flat connections over Riemann surfaces. Flat connections were studied by Narasimhan and Seshadri [NS65] from the point of view of holomorphic bundles. Let us give the main definitions of a principal bundle and connections. For more details on notions and results presented in this section, we refer the reader to [BM94; Ish99; Mor01].

Definition 3.2.13 (Principal bundle)**.** Let P and M be smooth manifolds and let G be a Lie group acting on P on the right, whith the action denoted by $(p,g)\mapsto pg$. The projection onto orbits is denoted by $\rho:P\to P/G$. A fiber bundle $\xi=P\xrightarrow{\pi}M$ is a *principal bundle* with group G if

- G acts freely on P,
- ξ is isomorphic to the projection bundle $\rho:P\to P/G$,
- For any local trivialisation $\phi:U\times G\to\pi^{-1}(U)$, for every $u\in U$,

$$g\mapsto\phi(u,g)\in\pi^{-1}(U)\simeq G$$

  is a homomorphism.

From the isomorphism with the projection bundle and the free action of the group, the fibers $\pi^{-1}(x), x \in M$ are orbits of P under the action of G. Let $\mathfrak{g}$ denote the Lie algebra of the Lie group G. For our purposes, we will consider M to be a Riemann surface.

Definition 3.2.14 (Connection as $\mathfrak{g}$-valued one form)**.** A *connection* is a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(M) \otimes \mathfrak{g}$. The *curvature form* $F : \Omega^1(M) \otimes \mathfrak{g} \to \Omega^2(M) \otimes \mathfrak{g}$ is defined as

$$F(\omega) = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega].$$

A connection ω is called *flat* if $F(\omega) = 0$. An equivalent description of a connection on a principal bundle is given by *equivariant paths liftings*, see [Ish99].

Definition 3.2.15 (Equivariant path lifting)**.** Let $\xi = P \xrightarrow{\pi} M$ be a principal bundle with group G and let $I = [0, 1]$. An *equivariant path lifting* associates to each smooth curve $\alpha : I \to M$ and $p \in \pi^{-1}(\alpha(0))$, a curve $\tilde{\alpha}_p : I' \subset I \to P$ such that

- $\pi \circ \tilde{\alpha}_p = \alpha$,
- $\forall g \in G, \forall t \in I' : \tilde{\alpha}_{pg}(t) = \tilde{\alpha}_p(t)g$.

Consider a curve $\alpha : I \to M$ such that $\alpha(0) = \alpha(1)$. A connection, or equivalently, an equivariant path lifting, lifts the curve α to $\tilde{\alpha}$. Let $p \in P$ such that $\pi(p) = \alpha(0)$. Since $p' = \tilde{\alpha}(1)$ lies in the same fiber as p and since the action of G is free, there exists $g \in G$, called the *holonomy* of the curve α such that $p' = pg$. As the fibers are orbits of the G-action, holonomies at two different base points p_1 and p_2 in the fiber of $\alpha(0)$ are conjugate. For a connection ω and a curve α, denote by $\mathrm{hol}_\omega(\alpha) \in G$ its holonomy.
Up to conjugation, the holonomy is defined for each curve. Flatness of a connection implies that the holonomy of a curve only depends on its homotopy class.

The space of connections on a compact, oriented surface is an infinite dimensional space. There is an action, called *gauge action* of the space $\mathcal{G} = \mathcal{C}^\infty(M, \mathfrak{g})$ on the space of flat connections on M denoted by $\mathcal{A}_M$ which preserves the curvature form F. We refer to [Mor01] for details on this action. Once quotiented by this action, the space of flat connections becomes a finite dimensional manifold. Via holonomies, this quotient space is isomorphic to the space of homomorphisms from the fundamental group of the surface to the group G up to conjugation.

Theorem 3.2.16 (Moduli space of flat connections)**.** *The map*

$$\begin{aligned} \mathcal{A}_M/\mathcal{G} &\to \mathrm{H}om(\pi_1(M), G)/G \\ \omega &\mapsto \mathrm{hol}_\omega(\cdot) \end{aligned}$$

is an isomorphism.

When M is the Riemann sphere with $\ell \geqslant 1$ boundary components, its fundamental group $\pi_1(M)$ admits the presentation

$$\pi_1(M) = \langle c_1, \ldots, c_\ell \mid c_1 \cdots c_\ell = 1 \rangle.$$

Therefore, for $G = \mathrm{SU}(n)$, the space of flat connections on $M = \mathbb{P}^1 \setminus \{p_1, \ldots, p_\ell\}$ with holonomies in conjugacy classes $\theta_1, \ldots, \theta_\ell$ for loops around punctured points is isomorphic to $\mathcal{M}(\theta_1, \ldots, \theta_\ell)$ defined in (3.2.1):

$$\{\omega \in \mathcal{A}_M \mid \mathrm{hol}_\omega(\alpha_i) \in \mathcal{O}(\theta_i), 1 \leqslant i \leqslant \ell\}/\mathcal{G} \simeq \mathcal{M}(\theta_1, \ldots, \theta_\ell) \tag{3.2.3}$$

Volume of moduli space of flat connections

The space of flat connections modulo the gauge action (3.2.3) admits a symplectic structure. This was established by Goldman [Gol84] from the perspective of fundamental group representations, and by Atiyah and Bott [AB83]. Atiyah, Bott, and Witten [Wit91] provided formulas for its symplectic volume, expressed as a sum over characters indexed by the irreducible representations of the group G. Witten used formulas established by Verlinde [Ver88], via a different approach, to compute explicit volumes in the case where the surface is the Riemann sphere and $G = \mathrm{SU}(2)$. Volumes of moduli space of flat connections are related to the Yang-Mills measure [Lév00] in the small surface regime, see [For93].

Any orientable Riemann surface of genus $g \geqslant 0$ with ℓ holes can be decomposed into $2g - 2 + \ell$ three-holed spheres. Therefore, the volume for more general surfaces can be derived from that of the three holed-sphere via a gluing procedure, see [MW99; Wit92]. Computing the volume of flat $\mathrm{SU}(n)$ connections on the three-holed sphere is therefore a first step towards volume computations for arbitrary Riemann surfaces. Chapter 6 presents a formula for this volume as a positive sum of polytope volumes.

3.3 Contributions to the subject

In this section, we present our results related to products of random unitary matrices, corresponding to the articles [FT24] and [Fra24] respectively presented in Chapters 6 and 7 of this thesis.

3.3.1 A positive density formula

In this section, we describe our results on a probabilistic version of the unitary Horn problem from Section 3.2. This can be seen as an analogue to the probabilistic version of the Hermitian Horn problem presented in Section 3.1.4. The main result of Section 3.1.4, namely Theorem 3.1.27 from [CZ18], expresses the density of the sum of random Hermitian matrices as the volume of the hive polytope from Knutson and Tao [KT99]. In the work [FT24], we give an expression for the probability density of eigenvalues of a product of unitary matrices in a similar fashion as a sum of volumes of explicit polytopes.

The set of conjugacy classes of $\mathrm{U}(n)$ is homeomorphic to the quotient $\mathcal{H} = (\mathbb{R}^n/\mathbb{Z}^n)/S_n$, where the symmetric group S_n acts on $(\mathbb{R}^n/\mathbb{Z}^n)$ by permutation of the coordinates. This quotient space is described by the set of non-increasing sequences of $[0,1[^n$. For $\theta = (\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n) \in \mathcal{H}$, let us denote by $\mathcal{O}(\theta)$ the corresponding conjugacy class

$$\mathcal{O}(\theta) := \left\{ Ue^{2i\pi\theta}U^*,\, U \in \mathrm{U}(n) \right\}, \text{ where } e^{2i\pi\theta} = \begin{pmatrix} e^{2i\pi\theta_1} & 0 & \cdots & \\ 0 & e^{2i\pi\theta_2} & & \\ \vdots & & \ddots & \\ & & & e^{2i\pi\theta_n} \end{pmatrix}.$$

The product structure on $\mathrm{U}(n)$ translates into a convolution product

$$* : \mathcal{M}_1(\mathcal{H}) \times \mathcal{M}_1(\mathcal{H}) \to \mathcal{M}_1(\mathcal{H})$$

on the space of probability distributions on $\mathcal{H}$ such that for $\theta, \theta' \in \mathcal{H}$, $\delta_\theta * \delta_{\theta'}$ is the distribution of $p(U_\theta U_{\theta'})$, where U_θ (resp. $U_{\theta'}$) is sampled uniformly on $\mathcal{O}(\theta)$ (resp. $\mathcal{O}(\theta')$) and

$p : \mathrm{U}(n) \to \mathcal{H}$ maps an element of $\mathrm{U}(n)$ to its conjugacy class in $\mathcal{H}$.

Let us denote by $\mathcal{H}_{reg} = \{\theta \in \mathcal{H}, \theta_1 > \theta_2 > \ldots > \theta_n\}$ the set of regular conjugacy classes of $\mathrm{U}(n)$, namely the ones of maximal dimension in $\mathrm{U}(n)$. For $\alpha, \beta \in \mathcal{H}_{reg}$, $\delta_\alpha * \delta_\beta$ admits a density $\mathrm{d}\mathbb{P}[\cdot|\alpha, \beta]$ with respect to the Lebesgue measure on

$$\left\{\gamma \in \mathcal{H} \mid \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i - \sum_{i=1}^{n} \gamma_i \in \mathbb{N}\right\} .$$

The toric hive cones $\mathcal{C}_g$

The main result of [FT24] is a positive formula for $\mathrm{d}\mathbb{P}[\cdot|\alpha, \beta]$ in terms of the volume of polytopes similar to the hive model of Knutson and Tao [KT99]. For $0 \leqslant d \leqslant n$, define the *toric hive* $R_{d,n}$ as the set

$$R_{d,n} := \left\{(v_1, v_2) \in [\![0, n]\!]^2, d \leqslant v_1 + v_2 \leqslant n + d\right\},$$

which can be represented as a discrete hexagon through the map $(v_1, v_2) \mapsto v_1 + v_2 e^{i\pi/3}$, see Figure 3.11 for a particular case and its hexagonal representation.

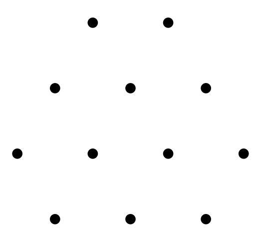

Figure 3.11: The set $R_{1,3}$ represented through the map $(v_1, v_2) \mapsto v_1 + v_2 e^{i\pi/3}$.

Boundary of the toric hive

For any set S and any function $f : R_{d,n} \to S$, we denote by f^A (resp f^B, f^C) the vector $(f((d-i) \vee 0, (n+d-i) \wedge n)_{0 \leqslant i \leqslant n}$ (resp. $(f(n+d-i \wedge n, i))_{0 \leqslant i \leqslant n}$, resp. $(f(n-i, i+d-n \vee 0))_{0 \leqslant i \leqslant n}$). The vectors f^A, f^B and f^C correspond respectively to the north-west, east and south-west boundaries of $R_{d,n}$ through the hexagonal representation, see Figure 3.12.

Toric rhombus concavity

Let us call a *lozenge* of $R_{d,n}$ any sequence $(v^1, v^2, v^3, v^4) \in (R_{d,n})^4$ corresponding to one of the three configurations of Figure 3.13 in the hexagonal representation (in which $|v^i - v^{i+1}| = 1$ for $1 \leqslant i \leqslant 3$).

Definition 3.3.1 (Regular labeling)**.** A function $g : R_{d,n} \to \mathbb{Z}_3$ is called a *regular labeling* whenever

- $g_i^A = n + i[3]$, $g_i^B = i[3]$ and $g_i^C = i[3]$,
- on any lozenge $\ell = (v^1, v^2, v^3, v^4)$,

$$\left(g(v^2) = g(v^4)\right) \Rightarrow \left\{g(v^1), g(v^3)\right\} = \left\{g(v^2) + 1, g(v^2) + 2\right\}.$$

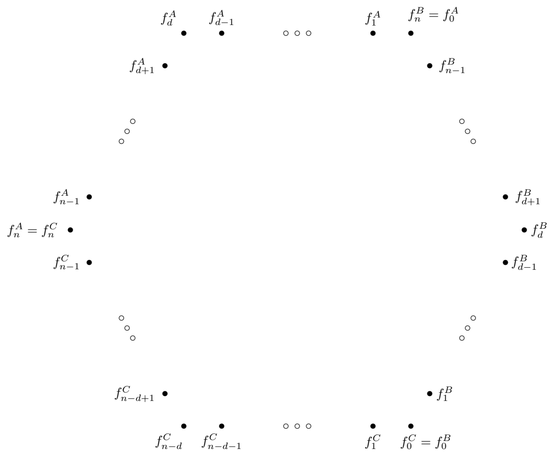

Figure 3.12: The set boundary vectors f^A, f^B and f^C.

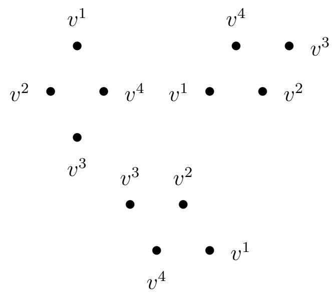

Figure 3.13: The three possible lozenges (v^1, v^2, v^3, v^4) (the position of the vertices can not be permuted).

A lozenge (v^1, v^2, v^3, v^4) for which $(g(v^1), g(v^2), g(v^3), g(v^4)) = (a, a+1, a+2, a+1)$ for some $a \in \{0, 1, 2\}$ is called *rigid*. The *support* of a regular labeling $g : R_{d,n} \to \mathbb{Z}_3$ is the subset $Supp(g) \subset R_{d,n}$ of vertices of $R_{d,n}$ which are not a vertex v_4 of a rigid lozenge (v^1, v^2, v^3, v^4).

An example of regular labeling is shown in Figure 3.14.

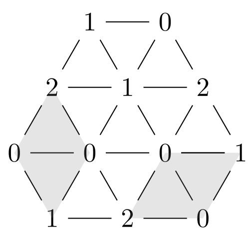

Figure 3.14: A regular labeling on $R_{d,n}$. Rigid lozenges are shaded.

Definition 3.3.2 (Toric hive cone)**.** A function $f : R_{d,n} \to \mathbb{R}$ is called *toric rhombus concave* with respect to a regular labeling $g : R_{d,n} \to \mathbb{Z}_3$ when $f(v_2) + f(v_4) \geqslant f(v_1) + f(v_3)$ on any lozenge $\ell = (v^1, v^2, v^3, v^4)$, with equality if ℓ is rigid with respect to g.

For any regular labeling g, the *toric hive cone* $\mathcal{C}_g$ with respect to g is the cone

$$\mathcal{C}_g = \left\{ f_{|Supp(g)} \mid f : R_{d,n} \to \mathbb{R} \text{ toric rhombus concave with respect to } g \right\}.$$

The hive cone from [KT99] is then a particular case of toric hive cone for $d = 0$. An example of a toric rhombus concave function in the case $n = 3, d = 1$ is given in Figure 3.15.

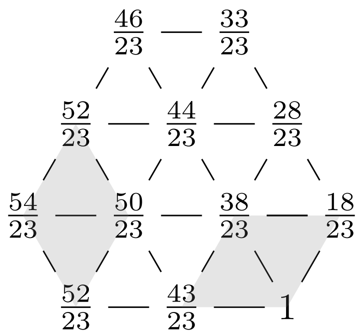

Figure 3.15: A toric rhombus convave function for $n = 3, d = 1$: shaded lozenge are the rigid ones yielding the equality cases in the toric rhombus concavity.

Definition 3.3.3 (Polytope $P^g_{\alpha,\beta,\gamma}$)**.** Let $n \geqslant 3$ and let $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ be such that $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ with $d \in \mathbb{N}$. Let g be a regular labeling on $R_{d,n}$. Then, $P^g_{\alpha,\beta,\gamma}$ is the polytope of $\mathbb{R}^{Supp(g)\setminus \partial R_{d,n}}$ consisting of functions in $\mathcal{C}_g$ such that

$$f^A = \left(\sum_{s=1}^n \beta_s + \sum_{s=1}^i \alpha_s \right)_{0 \leqslant i \leqslant n}, \; f^B = \left((d-i)^+ + \sum_{s=1}^i \beta_s \right)_{0 \leqslant i \leqslant n}, \; f^C = \left(d + \sum_{s=1}^i \gamma_s \right)_{0 \leqslant i \leqslant n}.$$

An example of an element of $P^g_{\alpha,\beta,\gamma}$ for $n = 3$ and $d = 1$ is depicted in Figure 3.15, for $\alpha = \left(\frac{13}{23} \geqslant \frac{6}{23} \geqslant \frac{2}{23} \right)$, $\beta = \left(\frac{18}{23} \geqslant \frac{10}{23} \geqslant \frac{5}{23} \right)$ and $\gamma = \left(\frac{20}{23} \geqslant \frac{9}{23} \geqslant \frac{2}{23} \right)$.

Our main result gives then a formula for the density of the convolution of regular conjugacy classes as a sum of volumes of polytopes coming from $\mathcal{C}_g$ for regular labeling g.

Theorem 3.3.4 (Probability density for product of conjugacy classes)**.** *Let $n \geqslant 3$ and let $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ be such that $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ with $d \in \mathbb{N}$. Then,*

$$d\mathbb{P}[\gamma|\alpha,\beta] = \frac{(2\pi)^{(n-1)(n-2)/2} \prod_{k=1}^{n-1} k! \Delta'(e^{2i\pi\gamma})}{n! \Delta'(e^{2i\pi\alpha}) \Delta'(e^{2i\pi\beta})} \sum_{g : R_{d,n} \to \mathbb{Z}_3 \text{ regular}} \mathrm{Vol}_g(P^g_{\alpha,\beta,\gamma}), \qquad (3.3.1)$$

where $\Delta'(e^{2i\pi\theta}) = 2^{n(n-1)/2} \prod_{i<j} \sin(\pi(\theta_i - \theta_j))$ for $\theta \in \mathcal{H}$ and Vol_g denotes the volume with respect to the Lebesgue measure on $\mathbb{R}^{Supp(g)\setminus \partial R_{d,n}}$.

As presented in (3.2.3) of Section 3.2.5, if we denote by $\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)$ the moduli space of flat $\mathrm{SU}(n)$-valued connections on the three holed-sphere Σ_0^3 for which the holonomies around a, b, c respectively belong to $\mathcal{O}(\alpha), \mathcal{O}(\beta)$ and $\mathcal{O}(\gamma)$, we have an isomorphism

$$\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma) \simeq \{(U_1, U_2, U_3) \in \mathcal{O}(\alpha) \times \mathcal{O}(\beta) \times \mathcal{O}(\gamma),\, U_1 U_2 U_3 = Id_{\mathrm{SU}(n)}\} / \mathrm{SU}(n).$$

As a corollary of Theorem 3.3.4, we get an expression of the volume of $\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)$ as a sum of volumes of explicit polytopes.

Corollary 3.3.5 (Volume of flat $\mathrm{SU}(n)$-connections on the sphere)**.** *Let $n \geqslant 3$ and consider the canonical volume form on* $\mathrm{SU}(n)$*. For $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ such that $|\alpha|_1$, $|\beta|_1$, $|\gamma|_1 \in \mathbb{N}$, then* $\mathrm{Vol}\left[\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)\right] \neq 0$ *only if $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i + \sum_{i=1}^n \gamma_i = n + d$ for some $d \in \mathbb{N}$, in which case*

$$\mathrm{Vol}\left[\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)\right] = \frac{2^{(n+1)[2]}(2\pi)^{(n-1)(n-2)}}{n!\Delta'(\mathrm{e}^{2i\pi\gamma})\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})} \sum_{g : R_{d,n} \to \mathbb{Z}_3 \text{ regular}} \mathrm{Vol}_g(P^g_{\alpha,\beta,\widetilde{\gamma}}),$$

where $\widetilde{\gamma} = (1 - \gamma_n, \dots, 1 - \gamma_1)$ and the polytopes $P^g_{\alpha,\beta,\widetilde{\gamma}}$ are defined in Definition 3.3.3.

3.3.2 Enumeration of crossings in two-step puzzles

In this section, we present our results corresponding to the article [Fra24] which is the subject of Chapter 7. The main result is Theorem 3.3.10 which counts configurations in two-step puzzles introduced in Section 3.2.4.

Definition 3.3.6 (Triangular lattice)**.** Let $n \geqslant 1$ and let $\xi = \mathrm{e}^{\frac{i\pi}{3}}$. Let us denote by $T_n = \{r + s\xi, 0 \leqslant r + s \leqslant n\}$ the vertices of the triangular lattice of size n and by $E_n = \{(x, x + v) \mid x, x + v \in T_n \text{ and } v \in \{-\xi^{2l}, 0 \leqslant l \leqslant 2\}\}$ the set of edges in T_n. The faces of the lattice T_n are triangles which are called direct (respectively reversed) if the corresponding vertices $(x_1, x_2, x_3) \in T_n^3$ can be labeled in such a way that $x_2 - x_1 = (1, 0)$ and $x_3 - x_1 = \xi$ (respectively $x_3 - x_1 = \overline{\xi}$).

Edges in E_n can only have three possible orientations. If $x = r + s\xi \in T_n$, we define three coordinates (x_0, x_1, x_2) by

$$x_0 \coloneqq n - (r + s),\ x_1 \coloneqq r \text{ and } x_2 \coloneqq s.$$

Definition 3.3.7 (Edge coordinate and type)**.** We say that an edge $e = (x, x + v)$ *is of type* l for $l \in \{0, 1, 2\}$ when $v = -\xi^{2l}$. The *origin* of e is x and the coordinates of e is the triple $(e_0, e_1, e_2) = (x_0, x_1, x_2)$. The *height* of e of type l is $h(e) = e_l$. Define also the boundary edges of E_n by

$$\begin{aligned}
\partial_0^{(n)} &\coloneqq (((n - r + 1, 0), (n - r, 0)), 1 \leqslant r \leqslant n) \\
\partial_1^{(n)} &\coloneqq \left((n\xi + (r - 1)\overline{\xi}, (n\xi + r\overline{\xi})), 1 \leqslant r \leqslant n\right) \\
\partial_2^{(n)} &\coloneqq \left(((r - 1)\overline{\xi}, r\overline{\xi}), 1 \leqslant r \leqslant n\right).
\end{aligned}$$

Definition 3.3.8 (Color map)**.** Let $n \geqslant 1$. A *color map* is a map $C : E_n \to \{0, 1, 3, m\}$ such that the boundary colors around each triangular face in the clockwise order is either $(0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 3)$ or $(0, 1, m)$ up to a cyclic rotation.

The values of a color map C on the boundary edges are denoted $\partial C = (\partial_0 C, \partial_1 C, \partial_2 C)$ and are defined for $l \in \{0, 1, 2\}$ as $\partial_l C = C_{|\partial_l^{(n)}}$. We say that C has boundary condition $\partial = (\partial_0, \partial_1, \partial_2)$ if $\partial C = \partial$.
Alternatively, one can view a color map C as a tiling of T_n by the set of edge labeled tiles of Figure 3.16 where tiles can be rotated. The last two tiles are respectively called 3 and m lozenges in accordance with the color of their middle edge.

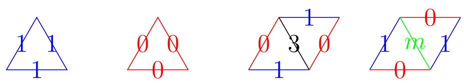

Figure 3.16: Possible tiles for color maps

As there is an equal number of both 0 and 1 labels on each side of two-step puzzles, we will consider boundary conditions $\partial C \in \{0,1\}^{3n}$ having an equal number of 0 and 1 colored edges respectively denoted by n_0 and n_1 so that $n_0 + n_1 = n$, see Figure 3.17 below. Such boundary conditions correspond to those of two-step puzzles [Buc+16], presented in Section 3.2.4, where one removed the labels 2 from the boundary 012 strings.

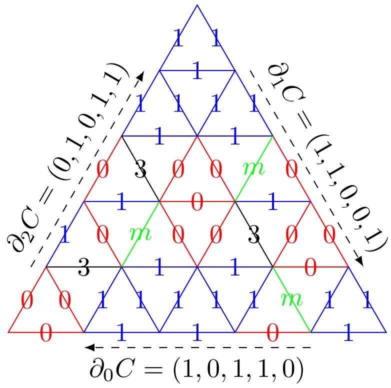

Figure 3.17: A color map on E_5 with boundary condition $\partial C = ((1,0,1,1,0),(1,1,0,0,1),(0,1,0,1,1))$.

Definition 3.3.9 (Gash numbers)**.** Let $C : E_n \to \{0,1,3,m\}$ be a color map. For any $l \in \{0,1,2\}$ and edge $e \in \partial_l^{(n)}$ denote by $n(C,e) = |\{e' \in \partial_l^{(n)} : h(e') < h(e) \text{ and } C(e') = 1\}|$ the number of 1 colored edges east (respectively north, south) to e if $e \in \partial_0^{(n)}$ (respectively $e \in \partial_1^{(n)}$, $e \in \partial_2^{(n)}$). The *gash numbers* of the color map C are defined for $l \in \{0,1,2\}$ as

$$G(C,l) := \sum_{e \in \partial_l^{(n)} : C(e)=0} n(C,e). \tag{3.3.2}$$

For instance, in the color map C of Figure 3.17, one has $G(C,0) = 4$, $G(C,1) = 4$, $G(C,2) = 1$. The main result of our work [Fra24] is Theorem 3.3.10 which gives a formula for the number of 3 and m colored edges in color maps which depends only on the gash numbers. The number of m colored edges in color maps is the number of rigid lozenges in a corresponding regular labeling which encodes equality conditions in polytopes appearing in Theorem 3.3.4, see Chapter 7 for details on this correspondance.

Theorem 3.3.10 (Label count in color maps)**.** *Let C be a color map on E_n having n_0, respectively n_1, edges of color 0, respectively 1, on each of its boundaries. Let $m(C)$ and $s(C)$ denote respectively the number of m and 3 colored edges in C. Then,*

$$m(C) = G(C,0) + G(C,1) + G(C,2) - n_0 n_1 \tag{3.3.3}$$

and

$$s(C) = 2n_0 n_1 - G(C,0) - G(C,1) - G(C,2). \tag{3.3.4}$$

3.4 Open questions

3.4.1 Volumes of flat connections on general surfaces

Corollary 3.3.5 provides a formula for the volume of flat $\mathrm{SU}(n)$-connections on the three-holed sphere. Such volumes are related to the Yang-Mills measure on Riemann surfaces in the small surface limit [For93], and it has been shown in [Wit92; MW99] that its computation for arbitrary compact Riemann surfaces can be reduced to the case of the three-punctured sphere by a sewing phenomenon. A similar inductive procedure is used in [Mir07] to reduce the volume problem for the moduli space of curves to the genus zero case. Understanding how the positive expression of Corollary 3.3.5 behaves when gluing three-holed spheres to form more general surfaces could lead to expressions for volumes of flat connections on general compact Riemann surfaces.

3.4.2 A two-step hive model

The work of Knutson and Tao [KT99; KT03] led to a description of Littlewood-Richardson coefficients as the number of integral points in the hive polytope, see Theorem 3.1.21. In addition to the puzzle rule for structure constants of the cohomology of the two-step flag variety of [Buc+16], one could ask for a description of these structure constants as the number of integral points in a polytope that would generalize the hive polytope of Knutson and Tao.

3.4.3 Extension to other Lie groups

For the Hermitian version of Section 3.1, extensions to real symmetric, quaternionic Hermitian were considered, see [Ful00] and [CMZ19]. Further extensions were also considered in [Par23] for noncompact reductive Lie groups and in [CM23] for compact Lie groups. One can ask for similar extensions regarding multiplicative versions in other Lie groups G which differ from the unitary group $\mathrm{U}(n)$. In particular, one could aim to derive the density for a product of permutation matrices, or orthogonal matrices. The case of matrices in $\mathrm{GL}(n)$ has been studied in [KO24].

Part II

Articles on characteristic polynomials

Chapter 4

Characteristic polynomial of Gaussian elliptic matrices

4.1 Introduction and main result

4.1.1 The model of the elliptic Ginibre Ensemble (EGE)

The random matrices that we consider in this chapter are sampled from the complex elliptic Ginibre Ensemble introduced by Girko in [Gir86]. This model is parametrized by $t \in [0,1]$ and interpolates between the Ginibre Ensemble and the Gaussian Unitary Ensemble (GUE) for $t = 0$ and $t = 1$ respectively. A concise review of this model can be found in [KS15]. Its law is the one of a random matrix given by the following construction.

Recall that a matrix sampled from the Gaussian Unitary Ensemble is a Hermitian random matrix whose density is proportional to $e^{-\mathrm{Tr}(M^2)/2}$. Consider X_n and Y_n independent random matrices sampled from the Gaussian Unitary Ensemble of size $n \geqslant 1$. The law of the elliptic Ginibre Ensemble at $t \in [0,1]$ is the law of the matrix

$$A_{n,t} = \sqrt{\frac{1+t}{2}} X_n + i\sqrt{\frac{1-t}{2}} Y_n, \tag{4.1.1}$$

where i is the imaginary unit. Equivalently, $A_{n,t}$ has a law proportional to

$$\exp\left(-\frac{1}{1-t^2}\mathrm{Tr}\left[M^*M - \frac{t}{2}(M^2 + (M^*)^2)\right]\right) \mathrm{d}M, \tag{4.1.2}$$

where $\mathrm{d}M = \prod_{1\leqslant i,j\leqslant n} \mathrm{d}M_{ij}$ is the product Lebesgue measure on the entries of the matrix, see [ADM23, Eq. (4)]. Notice that $A_{n,t}$ could also be defined as a centered complex Gaussian matrix whose entries a_{ij} satisfy $\mathbb{E}[|a_{ij}|^2] = 1$, $\mathbb{E}[a_{ij}a_{ji}] = t$ for every i,j, and for $i \neq j$ $\mathbb{E}[a_{ij}\bar{a}_{ji}] = \mathbb{E}[a_{ij}^2] = 0$ while covariance between a_{ij} and $a_{i'j'}$ are zero if $\{i,j\} \neq \{i',j'\}$. Moreover, for $i \neq j$, $\mathbb{E}[(a_{ij}a_{ji} - t)^2] = t^2$ and $\mathbb{E}[|a_{ij}a_{ji} - t|^2] = 1$. Many results are known for EGE matrices. In particular, the limiting eigenvalue distribution has been proved by Girko to be the uniform law on the ellipse centered at the origin with half long axis $1+t$ and short axis $1-t$. We refer to [Gir86, Theorem 7] and [Som+88] for the first instances of this result.

In the recent work [BCG22], it has been proved that the spectral radius of matrices with i.i.d. centered entries, called Girko matrices, converges in probability to 1 under the minimal assumption of a second moment on its entries. In order to derive this result, the authors considered the reciprocal characteristic polynomial associated to such matrices defined by $q_n(z) = z^n p_n\left(\frac{1}{z}\right)$ for $z \in \mathbb{D} = \{y \in \mathbb{C} : |y| < 1\}$, where p_n is the characteristic polynomial. The main result of [BCG22] is the convergence in law, for the topology of local uniform convergence, of the sequence of functions $\{q_n\}_{n\geqslant 1}$ to a random function which is universal, in the sense that its expression involves only the second moment of the entries of the matrix. Our result aims at deriving the convergence of the normalised characteristic polynomial in the case of the EGE (4.1.1) at each $t \in [0, 1]$ and at identifying the limiting object in the conjectured universality. In particular, for $t = 1$ our result gives the convergence of the characteristic polynomial for GUE matrices to a random analytic function.

Characteristic polynomials of random matrices have been studied extensively. For Haar unitary matrices or, more generally, for Circular β-Ensembles (CβE), the characteristic polynomial outside the unit disk behaves in a similar way as in Theorem 4.1.1 for $t = 0$ as is stated in [NPS23, Theorem 1.3]. Moreover, the characteristic polynomial inside and outside the unit disk exhibit the same but independent limiting behavior. More interestingly, the scaling limit around a point at the unit circle has been studied in [CNN17] by showing a convergence towards a random analytic function whose zeros form a determinantal point process on the real line. Limit expressions for the characteristic polynomial of CβE matrices are furthermore related to the Gaussian multiplicative chaos and to the theory of orthogonal polynomials on the unit circle, see [LN24]. In the case of Gaussian β-Ensembles, approximations of the characteristic polynomial in the complex plane were found in terms of log-correlated Gaussian fields, see [LP23]. For Haar random matrices, asymptotics for moments of derivatives of the characteristic polynomial have also been computed in [SW24]. They derive limits both inside the unit disk and for mesoscopic and microscopic regimes when z approaches the unit circle. The cases of orthogonal, symplectic and GUE random matrices have been studied in [Chh+19] where ratios of characteristic polynomials are shown to converge to a random entire function which was constructed in [CNN17] and related to Haar random matrices.

The study of the reciprocal characteristic polynomial for Girko matrices in [BCG22] was partially inspired from the work [BZ20] on Toeplitz matrices. The same object was studied for other models. The case of sparse matrix models having i.i.d. non-centered Bernoulli entries was treated in [Cos23]. The reciprocal characteristic polynomial of such matrices converges to a random function expressed using Poisson series [Cos23]. In [CLZ24], the same type of convergence was obtained for sums of random uniform permutation matrices. For a fixed number of random matrices in the sum, the limit has the same form given by the exponential of a Poisson series, whereas for a number of terms going to infinity in a prescribed way, the limit has the form given by the exponential of a Gaussian series as in [BCG22]. Exponential of Poisson series were also identified to be the limit of characteristic polynomials for permutation matrices in [Bah19a; Bah19b].

In relation to elliptic matrices, the uniform law on the ellipse can be obtained as the asymptotic distribution of zeroes of random polynomials which are related to Weyl polynomials, see [Bri+24].

The motivation from the work [BCG22] was to obtain the convergence of the spectral radius for Girko matrices. One could ask for a study of the fluctuations around the limit. For the Ginibre Ensemble, one has Gumbel fluctuations for the maximum modulus around 1, see [Rid03]. The Gumbel distribution also appears as the limit fluctuation for the largest real part of either real or complex Ginibre matrices [Cip+22]. For the GUE, one has Tracy-Widom fluctuations for the maximum eigenvalue around 2, see [TW02] and references therein. In [Joh07], we may find a family of determinantal processes that interpolates between a Poisson process with intensity e^{-x} and the Airy process. The distribution function of its last particle interpolates between the Gumbel and Tracy-Widom distributions, see [Joh07, Theorem 1.3]. As a two-dimensional version, [Ben10] considered the elliptic Ginibre Ensemble and an interpolating determinantal process to prove scaling limits for the eigenvalue point process.

4.1.2 Main result

Let $n \geqslant 1$ and $t \in [0,1]$. Consider $p_{n,t}(z) = \det(z - \frac{1}{\sqrt{n}} A_{n,t})$, the characteristic polynomial scaled by $\frac{1}{\sqrt{n}}$ of a matrix $A_{n,t}$ sampled from the elliptic Ginibre Ensemble (4.1.1). Define $f_{n,t} : \mathbb{D} \to \mathbb{C}$ as the normalised characteristic polynomial of $A_{n,t}$,

$$f_{n,t}(z) := \det\left(1 + tz^2 - \frac{z}{\sqrt{n}} A_{n,t}\right) \mathrm{e}^{-\frac{ntz^2}{2}}. \tag{4.1.3}$$

We endow the space of holomorphic functions on $\mathbb{D}$ with the topology of uniform convergence on compact sets and state our main result as follows.

Theorem 4.1.1 (Convergence of the normalised characteristic polynomial)**.** *We have the convergence in law, for the topology of local uniform convergence,*

$$f_{n,t} \xrightarrow[n\to\infty]{\text{law}} \exp(-F_t)$$

where F_t is the Gaussian holomorphic function on $\mathbb{D}$ defined by

$$F_t(z) := \sum_{k \geqslant 1} X_k \frac{z^k}{\sqrt{k}} \tag{4.1.4}$$

for a family $(X_k)_{k\geqslant 1}$ of independent Gaussian random variables on $\mathbb{C}$ satisfying

$$\mathbb{E}[X_k] = 0, \ \mathbb{E}[X_k^2] = t^k \text{ and } \mathbb{E}[|X_k|^2] = 1.$$

Let us give some intuition for the choice in (4.1.3). Since the empirical measure of eigenvalues of $A_{n,t}$ converges to the uniform measure σ_t on the ellipse

$$\mathcal{E}_t = \left\{ x + iy \in \mathbb{C} : \left(\frac{x}{1+t}\right)^2 + \left(\frac{y}{1-t}\right)^2 \leqslant 1 \right\}, \tag{4.1.5}$$

to study the behavior of the characteristic polynomial $p_{n,t}$ on $\mathbb{C} \setminus \mathcal{E}_t$, we send this set to the unit disk in the simplest holomorphic way, namely, by using the map $g_t : \mathbb{D} \setminus \{0\} \to \mathbb{C} \setminus \mathcal{E}_t$,

$$g_t(z) = \frac{1}{z} + tz.$$

In the case of $t = 1$, we should define $\mathcal{E}_1 = [-2, 2]$ and $\sigma_1 = \lim_{t\to 1} \sigma_t$ is the semi-circular law, which is consistent with Wigner's semicircular law. In this case, g_1 is the so-called Joukowsky transform and, moreover, we have the simple relation $g_t(z) = \sqrt{t} g_1(\sqrt{t} z)$ whenever $t \neq 0$. Under this change of variables, the characteristic polynomial $p_{n,t}$ is

$$p_{n,t} \circ g_t(z) = \det\left(g_t(z) - \frac{A_{n,t}}{\sqrt{n}}\right) = \frac{1}{z^n} \det\left(1 + tz^2 - \frac{z}{\sqrt{n}} A_{n,t}\right)$$

Finally, $n^{-1} \log \det \left(1 + tz^2 - zn^{-1/2} A_{n,t}\right)$ should converge to $\int \log(1 + tz^2 - zw) \mathrm{d}\sigma_t(w)$ which is $tz^2/2$, as seen in the proof of Lemma 4.2.14, which suggests the exponential factor in the expression of $f_{n,t}$.

Using these notations, from Theorem 4.1.1 we obtain the convergence of the normalised characteristic polynomial $\tilde{p}_{n,t}(u) = (g_t^{-1}(u))^n \mathrm{e}^{-nt(g_t^{-1}(u))^2/2} p_{n,t}(u)$

$$\tilde{p}_{n,t} \xrightarrow{\text{law}} \exp(-F_t \circ g_t^{-1})$$

for the topology of uniform convergence on compact sets of $\mathbb{C} \setminus \mathcal{E}_t$. This is, in fact, equivalent to Theorem 4.1.1 due to the holomorphicity of $f_{n,t}$ at zero. It explains the notation "normalized characteristic polynomial" since $f_{n,t}$ and $\tilde{p}_{n,t}$ are the same functions in different coordinate systems.

From Theorem 4.1.1, one can derive the following result given in [OR14, Theorem 2.2] for a class of elliptic matrices that includes our Gaussian case. Nevertheless, since an explicit density can be written for the eigenvalues in the Gaussian case, we may also use large deviation arguments to obtain the lack of outliers.

Corollary 4.1.2 (Lack of outliers)**.** *Let $C \subset \mathbb{C}$ be a closed set disjoint from $\mathcal{E}_t$. Then,*

$$N_n(C) := \#\left\{i \in [n] : \frac{\lambda_i}{\sqrt{n}} \in C\right\} \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{4.1.6}$$

Proof. Let $C \subset \mathbb{C}$ be a closed set disjoint from $\mathcal{E}_t$. Recall that $g_t(z) = \frac{1}{z} + tz$ and consider $\tilde{C} = g_t^{-1}(C)$ which is closed in $\mathbb{D} \setminus \{0\}$ so that its closure K on $\mathbb{D}$ is compact. Then,

$$\begin{aligned}
\mathbb{P}[|N_n(C)| > 0] &= \mathbb{P}[\inf_{z\in C} |p_{n,t}(z)| = 0] \\
&= \mathbb{P}[\inf_{u\in K} |f_{n,t}(u)| = 0] \\
&\to \mathbb{P}[\inf_{u\in K} |e^{-F_t(u)}| = 0] = 0.
\end{aligned}$$

$\square$

In fact, we expect an analogue of Theorem 4.1.1 to hold in a much more general setting as conjectured in [BCG22]. The limit would only depend on some of the first four moments of the coefficients of the random matrix as suggested in Section 4.1.3. A glimpse of this universality can be seen, for instance, when calculating the expected value of the characteristic polynomial. This depends only on $t = \mathbb{E}[a_{12}a_{21}]$ as explained in the proof of the following theorem where a simple expression for its limit is stated.

Theorem 4.1.3 (Average characteristic polynomial)**.** *For each n, let $A_{n,t} = (a_{ij}, 1 \leqslant i, j \leqslant n)$ be a random matrix such that $\{(a_{ij}, a_{ji}), 1 \leqslant i < j \leqslant n\}$ are i.i.d. centered pairs which are independent of the i.i.d. centered family $\{a_{ii}, 1 \leqslant i \leqslant n\}$ with $\mathbb{E}[|a_{ij}|^2] < \infty$ for all $1 \leqslant i, j \leqslant n$ and $\mathbb{E}[a_{12}a_{21}] = t \in [0, 1]$. Then, for z uniformly in $\mathbb{D}$,*

$$\lim_{n\to+\infty} \mathbb{E}\left[\det\left(1 + tz^2 - \frac{z}{\sqrt{n}} A_{n,t}\right) \mathrm{e}^{-\frac{ntz^2}{2}}\right] = \frac{1}{\sqrt{1 - tz^2}}. \tag{4.1.7}$$

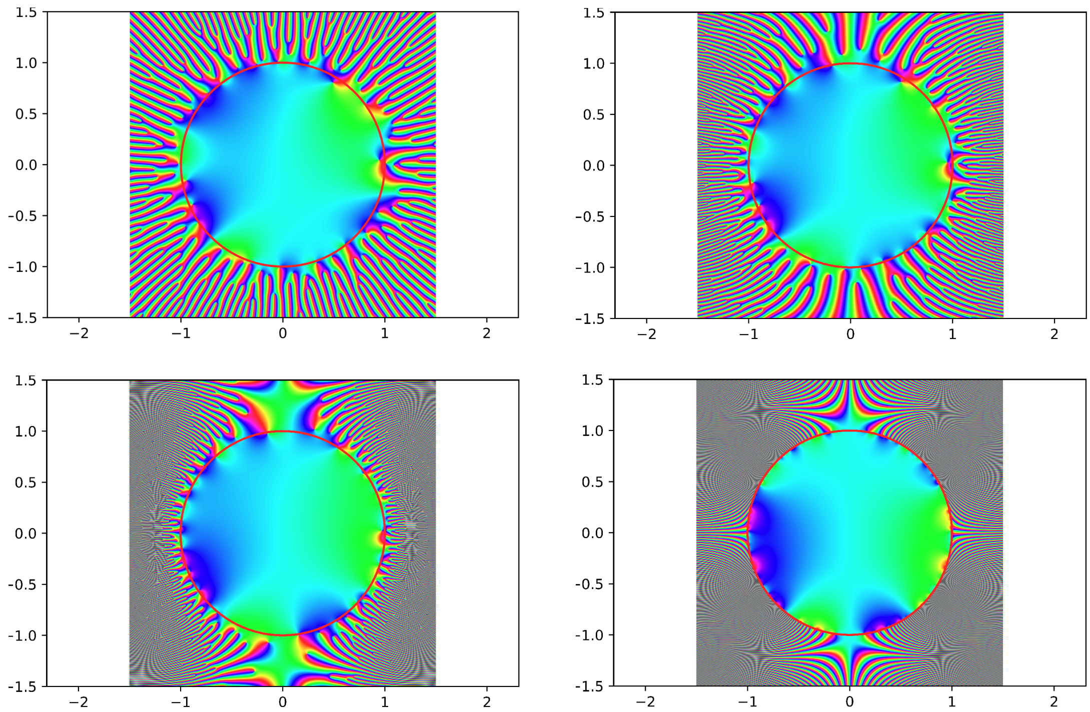

Figure 4.1: Illustration of Theorem 4.1.1. Phase portrait of the normalised characteristic polynomial of an EGE matrix of size 250 for different values of t: 0 (top left), 0.3 (top right), 0.6 (bottom left) and 1 (bottom right). The unit circle is represented in red.

4.1.3 Open questions and comments

Extension via matching moments

Since the way we show tightness is by controlling the second moment of $f_{n,t}$ and since this second moment depends only on the first four moments of $A_{n,t}$, tightness of $f_{n,t}$ still holds for the model described in Section 4.1.3 for coefficients $(a_{ij})_{i,j\geqslant 1}$ whose first four moments coincide with those of the EGE. Moreover, the proof of convergence of the coefficients of $f_{n,t}$ also works in the case where the coefficients have all moments finite so that Theorem 4.1.1 holds for coefficients $(a_{ij})_{i,j\geqslant 1}$ with all moments finite and whose first four moments coincide with those of the EGE.

Minimal moment condition and universality

As conjectured in [BCG22], the convergence in Theorem 4.1.1 of the normalised characteristic polynomial is believed to hold under the minimal moment condition

$$\mathbb{E}\left[|a_{12}a_{21}|^2\right] < \infty \tag{4.1.8}$$

on the entries $(a_{ij})_{i,j\geqslant 1}$, which gives a condition of a fourth order moment for Wigner matrices and second order moment for Girko matrices. The context adapted to this conjecture is the one of elliptic random matrices [NO15, Definition 1.3]. This model was introduced by Girko in [Gir86] and [Gir95]. A version of this model consists of the following matrices. Consider a family $(a_{ij})_{i,j\geqslant 1}$ of square-integrable centered random variables such that $\{(a_{ij}, a_{ji}) : i < j\} \cup \{a_{ii} : i \geqslant 1\}$ is an independent family of random elements and whose law is invariant under any permutation of the indices or, equivalently, the law of (a_{ij}, a_{ji}) coincides with the law of $(a_{i'j'}, a_{j'i'})$ whenever $|\{i,j\}| = |\{i',j'\}|$. If

$$\mathbb{E}[|a_{12}|^2] = 1 \quad \text{and} \quad \mathbb{E}[a_{12}a_{21}] = t,$$

the matrix $A_n = (a_{ij})_{1\leqslant i,j\leqslant n}$ is said to be t-Girko. The convergence of the average eigenvalue distribution towards the uniform distribution on the ellipse (4.1.5) has been proved under different conditions on the variables, see [NO15; OR14; Nau13]. We expect the following version of Theorem 4.1.1 to hold for the general t-Girko matrices described above. Denoting $\tau = \mathbb{E}[a_{12}^2]$, $s = \mathbb{E}[a_{11}^2] - t - \tau$ and $q = \mathbb{E}[(a_{12}a_{21} - t)^2] - t^2 - \tau^2$, the limit of $\det\left(1 + tz^2 - z\frac{A_{n,t}}{\sqrt{n}}\right)\exp(-ntz^2/2)$ is expected to be given by

$$\sqrt{1-\tau z^2}\mathrm{e}^{-sz^2/2}\mathrm{e}^{-qz^4/4}\mathrm{e}^{-\sum_{k\geqslant 1} Y_k \frac{z^k}{\sqrt{k}}} \tag{4.1.9}$$

where $(Y_k)_{k\geqslant 1}$ are independent centered complex Gaussians such that Y_1 has the same variance as a_{11}, Y_2 has the same variance as $a_{12}a_{21}$ and, for $k \geqslant 3$, the variance of Y_k is the sum of the k-th power of the variance of a_{12} and the k-th power of the covariance of a_{12} and a_{21} or, somewhat more explicitly, $\mathbb{E}[Y_k^2] = \mathbb{E}[a_{12}^2]^k + \mathbb{E}[a_{12}a_{21}]^k = \tau^k + t^k$ and $\mathbb{E}[|Y_k|^2] = \mathbb{E}[|a_{12}|^2]^k + \mathbb{E}[a_{12}\overline{a_{21}}]^k = 1 + \mathbb{E}[a_{12}\overline{a_{21}}]^k$.

Matrices with entries in $\{0,1\}$

As described above, a convergence of the reciprocal characteristic polynomial for matrices with independent Bernoulli entries with non-zero expectation has been proved in [Cos23]. The limiting random holomorphic function can be expressed using Poisson random variables, see [Cos23, Theorem 2.3]. One could ask for an analogue of the elliptic Ginibre Ensemble for such matrices and for the convergence of its normalised characteristic polynomial.

Extension of the work [CLZ24] for Ewens distributed random permutations can be found in [Fra25]. The convergence of traces of such random matrices were understood from the work of Nikeghbali and Zeindler [NZ13], while tightness uses asymptotics found by Hwang [Hwa94].

Determinantal Coulomb gases

As explained in 4.1.3, this work can be thought of as a first step towards the convergence of the characteristic polynomial outside the support of the equilibrium measure for general elliptic random matrices. Nevertheless, we could have followed a different path, which is to look the Elliptic Ginibre Ensembles as a particular case of a determinantal Coulomb gas. In this vein, it may be possible to show the convergence of the traces by adapting results from [AHM15] and to show tightness of the characteristic polynomial outside the support of the equilibrium measure for more general determinantal Coulomb gases by using, for instance, the results from [AC23].

4.2 Proof of Theorem 4.1.1

From now on, given that the case $t = 0$ is already treated in [BCG22], we assume $t \neq 0$ and omit the index t since it is considered fixed for the rest of the chapter.

As is standard, to show that $\{f_n\}_{n\geqslant 1}$ converges we show that $\{f_n\}_{n\geqslant 1}$ is tight and that the coefficients in its power-series expansion around the origin converge in law. We state this classical fact as Lemma 4.2.1 below and we refer to [BCG22, Section 4.2] for a proof. Recall that $\mathcal{H}(\mathbb{D})$ is the space of holomorphic functions on the unit disk $\mathbb{D}$ endowed with the topology of uniform convergence on compact sets.

Lemma 4.2.1 (Tightness and convergence of coefficients imply convergence of functions)**.** *Let $\{h_n\}_{n\geqslant 1}$ be a sequence of random elements in $\mathcal{H}(\mathbb{D})$ and denote the coefficients of h_n by $(\xi_k^{(n)})_{k\geqslant 0}$ so that for all $z\in\mathbb{D}$, $h_n(z)=\sum_{k\geqslant 0}\xi_k^{(n)}z^k$. Suppose also that the following conditions hold.*

- (a) *The sequence $\{h_n\}_{n\geqslant 1}$ is a tight sequence of random elements of $\mathcal{H}(\mathbb{D})$.*
- (b) *There exists a sequence $(\xi_k)_{k\geqslant 0}$ of random variables such that, for every $m\geqslant 0$, the vector $(\xi_0^{(n)},\dots,\xi_m^{(n)})$ converges in law as $n\to\infty$ to $(\xi_0,\dots,\xi_m)$.*

Then, $h(z)=\sum_{k\geqslant 0}\xi_k z^k$ is a well-defined function in $\mathcal{H}(\mathbb{D})$ and h_n converges in law towards h in $\mathcal{H}(\mathbb{D})$ for the topology of local uniform convergence.

In Lemma 4.2.1, there are two topologies involved, namely the topology of uniform convergence on compact subsets of the unit disk for the space $\mathcal{H}(\mathbb{D})$ and the weak topology, or the convergence in law, in the probability space. The first step is to show the following theorem.

Theorem 4.2.2 (Tightness)**.** *The sequence $\{f_n\}_{n\geqslant 1}$ is tight.*

The proof uses known properties of the elliptic Ginibre Ensemble and its relation to scaled Hermite polynomials. In particular, it relies on the determinantal aspect of its eigenvalue point process. A local uniform control is derived from [ADM23]. This control allows us to derive tightness thanks to Montel's theorem as stated in Lemma 4.2.5 below. This ensures that Condition (a) of Lemma 4.2.1 is satisfied for $\{f_n\}_{n\geqslant 1}$.

To guarantee Condition (b) of Lemma 4.2.1, we express the coefficients of $\{f_n\}_{n\geqslant 1}$ using a family of polynomials that we call the modified Chebyshev polynomials.

Definition 4.2.3 (Modified Chebyshev polynomials)**.** The *modified Chebyshev polynomials* are the polynomials $\{P_k\}_{k\geqslant 1}$ satisfying the recurrence relation

$$P_{k+1}=XP_k-tP_{k-1},\quad P_1=X,\quad P_2=X^2-2t. \tag{4.2.1}$$

We may also give the modified Chebyshev polynomials more explicitly by their coefficients

$$\alpha_{k-2j}^{(k)}:=(-t)^j\frac{k}{k-j}\binom{k-j}{j} \tag{4.2.2}$$

so that $P_k=\sum_{j\geqslant 0}\alpha_{k-2j}^{(k)}X^{k-2j}$ or by its generating function

$$\sum_{k\geqslant 1}P_k(w)\frac{z^k}{k}=-\log(1+tz^2-zw). \tag{4.2.3}$$

The latter expression can be obtained either by noticing that $P_k(w)=2\sqrt{t}^kT_k(w/(2\sqrt{t}))$ for T_k the classical Chebyshev polynomials of the first kind or by taking the derivative in z of (4.2.3), which is the rational expression $(w-2zt)(1+tz^2-zw)^{-1}$, so that

$(\sum_{k\geqslant 0} P_{k+1}(w)z^k)(1+tz^2-zw) = w-2zt$ which is the recurrence (4.2.1).

Then, for $z \in \mathbb{D}$ small enough using (4.2.3)

$$1+tz^2-\left(\frac{A_n}{\sqrt{n}}\right)z = \exp\left(-\sum_{k\geqslant 1} P_k\left(\frac{A_n}{\sqrt{n}}\right)\frac{z^k}{k}\right),$$

by, for instance, showing this for diagonalizable matrices and the using density of this set of matrices. Finally, taking the determinant we obtain that, for $z \in \mathbb{D}$ small enough,

$$f_n(z) := \det\left(1+tz^2-\frac{z}{\sqrt{n}}A_n\right)\mathrm{e}^{-\frac{ntz^2}{2}} \;=\; \exp\left(-\sum_{k\geqslant 1} U_k^{(n)}\frac{z^k}{k}\right), \tag{4.2.4}$$

where

$$U_k^{(n)} := \mathrm{Tr}\left[P_k\left(\frac{A_n}{\sqrt{n}}\right)\right] + nt\delta_{k=2}. \tag{4.2.5}$$

In particular, the first m coefficients of f_n can be expressed as polynomials of $U_0^{(n)}, \ldots, U_m^{(n)}$ which are independent of n and vice versa. Thus, showing the convergence in law of the variables $(U_k^{(n)})_{k\geqslant 1}$ is equivalent to showing the convergence in law of the coefficients of f_n, which is our way of characterizing the limit in Condition (b) of Lemma 4.2.1. Since it is easier to deal with traces, we will study $U_k^{(n)}$ and prove the convergence stated in the following theorem.

Theorem 4.2.4 (Convergence of the traces of Chebyshev polynomials)**.**

$$(U_k^{(n)})_{k\geqslant 1} \xrightarrow[n\to\infty]{\text{law}} (\sqrt{k}X_k)_{k\geqslant 1},$$

where $(X_k)_{k\geqslant 1}$ is a family of independent centered complex Gaussian random variables such that $\mathbb{E}[X_k^2] = t^k$ and $\mathbb{E}[|X_k|^2] = 1$.

Conclusion of the proof of Theorem 4.1.1. We use Lemma 4.2.1 with $h_n = f_n$ and ξ_k being the coefficients of $h(z) = e^{-F_t(z)} = \exp(-\sum_{k\geqslant 1} X_k \frac{z^k}{\sqrt{k}})$. Theorem 4.2.2 ensures that Condition (a) is satisfied while Theorem 4.2.4 ensures that Condition (b) is satisfied. □

4.2.1 Tightness: Proof of Theorem 4.2.2

Recall that $f_n : \mathbb{D} \to \mathbb{C}$ is given by

$$f_n(z) = \det\left(1+tz^2-\frac{z}{\sqrt{n}}A_n\right)\mathrm{e}^{-\frac{ntz^2}{2}}.$$

and that the space $\mathcal{H}(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$ is endowed with the topology of uniform convergence on compact sets, whereas the probability space is endowed with the weak convergence. Lemma 4.2.5 below is the stochastic version of Montel's theorem which reduces the proof of tightness to a control on compact sets.

Lemma 4.2.5 (Montel's theorem)**.** *Suppose that for every compact $K \subset \mathbb{D}$, the sequence $(||f_n||_K)_{n\geqslant 1}$ is tight, where $||f_n||_K = \max_{z\in K}|f_n(z)|$. Then, $\{f_n\}_{n\geqslant 1}$ is tight.*

Proof. It is a consequence of the classical Montel's theorem of complex analysis. See, for instance, [Shi12, Proposition 2.5]. □

Remark 4.2.6. By the subharmonicity of $|f_n(z)|^2$, saying that $(\mathbb{E}[\|f_n\|_K^2])_{n\geqslant 1}$ is a bounded sequence for every compact $K \subset \mathbb{D}$ is equivalent to saying that $(\sup_{z\in K} \mathbb{E}[|f_n(z)|^2])_{n\geqslant 1}$ is a bounded sequence for every compact $K \subset \mathbb{D}$. See, for instance, [Shi12, Lemma 2.6]. In the Girko case of [BCG22], one had a remarkable orthogonality of the sub-determinants which led to an upper bound on the desired quantity. As we no longer have this property, our proof is based on [AV03] which exploits the integrability of the elliptic Ginibre Ensemble.

Our goal is to control $\mathbb{E}[|f_n(z)|^2]$ and, for this, we will begin by giving an explicit expression of the second moment using Hermite polynomials.

Definition 4.2.7 (Hermite polynomials)**.** The *Hermite polynomials* $\{H_n\}_{n\geqslant 0}$ are the monic orthogonal polynomials with respect to the measure $\mathrm{e}^{-x^2/2}\mathrm{d}x$ on $\mathbb{R}$ so that

$$\int_{\mathbb{R}} H_n(x)H_m(x)\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x = \sqrt{2\pi}n!\delta_{n,m}.$$

Recall that $g_t(z) = z^{-1} + tz$.

Lemma 4.2.8 (Hermite expression of the characteristic polynomial)**.** *For $n \geqslant 1$ and any $z \in \mathbb{D}\setminus\{0\}$, one has the following expression*

$$\mathbb{E}[|f_n(z)|^2] = \frac{n!|z|^{2n}}{n^n}\left|\mathrm{e}^{-ntz^2}\right|\sum_{k=0}^{n}\frac{t^k}{k!}\left|H_k\left(\sqrt{\frac{n}{t}}g_t(z)\right)\right|^2. \tag{4.2.6}$$

Proof. In the case of the elliptic Ginibre Ensemble given by (4.1.1), the matrix $A_{n,t}$ has the following density, which can be found in [ADM23, eq. (4)].

$$\mathrm{d}\mathbb{P}_t(M) = \left(\frac{1}{\pi\sqrt{1-t^2}}\right)^{n^2}\exp\left(-\frac{1}{1-t^2}\mathrm{Tr}\left[MM^* - \frac{t}{2}(M^2+(M^*)^2)\right]\right)\mathrm{d}M \tag{4.2.7}$$

which is associated to the weight function

$$\begin{aligned} w_t(z) &= \frac{1}{\pi\sqrt{1-t^2}}\exp\left(-\frac{1}{1-t^2}\left(|z|^2 - \frac{t}{2}(z^2+\overline{z}^2)\right)\right) \\ &= \frac{1}{\pi\sqrt{1-t^2}}\exp\left(-\left(\frac{x^2}{1+t}+\frac{y^2}{1-t}\right)\right) \end{aligned}$$

with $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$. In order to use the main theorem of [AV03], we should compute the orthonormal polynomials with respect to $w_t(z)\mathrm{d}z$. Using [ADM23, eq. (3)], these polynomials are $\{P_n\}_{n\geqslant 0}$ given by

$$P_n(z) = \frac{\sqrt{t^n}}{\sqrt{n!}}H_n\left(\frac{z}{\sqrt{t}}\right). \tag{4.2.8}$$

For M sampled from (4.2.7) and any $u, v \in \mathbb{C}$, we may use [AV03, eq. (2.11)] to get

$$\mathbb{E}\left[\det(u-M)\overline{\det(v-M)}\right] = n!\sum_{k=0}^{n}P_k(u)\overline{P_k(v)}, \tag{4.2.9}$$

where the global factor $n!$ is the square inverse of the dominant coefficient of P_n. Finally, setting $u = v = g_t(z) = z^{-1} + tz$ gives

$$\begin{aligned} \mathbb{E}[|f_n(z)|^2] &= \mathbb{E}\left[\left|\mathrm{e}^{-\frac{ntz^2}{2}}\left(\frac{z}{\sqrt{n}}\right)^n\det\left(\sqrt{n}(z^{-1}+tz) - A_{n,t}\right)\right|^2\right] \\ &= \frac{n!|z|^{2n}}{n^n}\left|\mathrm{e}^{-ntz^2}\right|\sum_{k=0}^{n}\frac{t^k}{k!}\left|H_k\left(\sqrt{\frac{n}{t}}g_t(z)\right)\right|^2 \end{aligned}$$

which is the desired expression of $\mathbb{E}[|f_{n,t}(z)|^2]$ in terms of Hermite polynomials.

□

With the help of the expression (4.2.6) and using the results from [ADM23], we will control $\mathbb{E}[|f_n(z)|^2]$ uniformly on bounded sets. In fact, [ADM23] allows us to give an explicit expression for the limit of $\mathbb{E}[|f_n(z)|^2]$. Since we do not need an explicit expression, we will only state the following.

Lemma 4.2.9 (Convergence of the second moment)**.** *There exists a continuous function $\mathcal{F} : \mathbb{D} \setminus \{0\} \to (0, \infty)$ such that, uniformly on compact sets,*

$$\mathbb{E}[|f_n|^2] \xrightarrow[n\to\infty]{} \mathcal{F}.$$

Since f_n is holomorphic on the whole disk $\mathbb{D}$, one can extend the control on any disk $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ for $r \in (0, 1)$. This is written in the next proposition.

Proposition 4.2.10 (Uniform control)**.** *For every $r \in (0, 1)$ there exists $C_r > 0$ such that*

$$\mathbb{E}[\|f_n\|^2_{\mathbb{D}_r}] \leqslant C_r \text{ for every } n \geqslant 1.$$

Proof. By Lemma 4.2.9, we have a bound for $\mathbb{E}[|f_{n,t}(z)|^2]$ on compact sets of $\mathbb{D}\setminus\{0\}$. This is the same as a bound for $\mathbb{E}[\|f_{n,t}\|^2_K]$ for compact sets $K \subset \mathbb{D} \setminus \{0\}$ by Remark 4.2.6. We may obtain a bound for $\mathbb{E}[\|f_{n,t}\|^2_{\mathbb{D}_r}]$ for $r \in (0, 1)$ by using that $\|f_{n,t}\|_{\mathbb{D}_r} \leqslant \|f_{n,t}\|_{\partial\mathbb{D}_r}$ thanks to the maximum modulus principle. □

Proof of Lemma 4.2.9. Recall the function $g_t : \mathbb{D} \to \mathbb{C} \setminus \mathcal{E}_t$ given by $g_t(z) = \frac{1}{z} + tz$ and define $L_n : \mathbb{C} \setminus \mathcal{E}_t \to [0, \infty)$ by

$$L_n(u) = \sum_{k=0}^{n-1} \frac{t^k}{k!} \left| H_k\left(\sqrt{\frac{n}{t}}u\right)\right|^2.$$

By using the contour integral representation around a small loop enclosing the origin,

$$L_n(u) = \frac{1}{2\pi i} \oint_0 \frac{e^{nF_u(s)}}{t-s} \frac{ds}{\sqrt{1-s^2}}, \quad \text{with} \quad F_u(s) = \frac{s}{t}\left(\frac{\mathrm{Re}(u)^2}{1+s} + \frac{\mathrm{Im}(u)^2}{1-s}\right) - \log s + \log t,$$

the following has been proved in [ADM23, Theorem II.12, (i)] and [ADM23, Theorem II.13, (i)] for $u \in \mathbb{C} \setminus \mathcal{E}_t$ and $z = g_t^{-1}(u)$,

$$L_n(u) = \frac{1}{2\pi}\sqrt{\frac{2\pi}{nF_u''(t|z|^2)}} \frac{e^{nF_u(t|z|^2)}}{\sqrt{1-t^2|z|^4}} \frac{1}{t(1-|z|^2)} \left(1 + O\left(\frac{1}{n}\right)\right), \tag{4.2.10}$$

where the error term is uniform on compact sets of $\mathbb{C}\setminus\mathcal{E}_t$. Here $s = t|z|^2$ is a critical point of F_u which can be seen by first calculating

$$\begin{aligned} tF_u'(s) &= \frac{\mathrm{Re}(u)^2}{1+s} + \frac{\mathrm{Im}(u)^2}{1-s} + s\Big(-\frac{\mathrm{Re}(u)^2}{(1+s)^2} + \frac{\mathrm{Im}(u)^2}{(1-s)^2}\Big) - \frac{t}{s} \\ &= \frac{\mathrm{Re}(u)^2}{(1+s)^2} + \frac{\mathrm{Im}(u)^2}{(1-s)^2} - \frac{t}{s} \\ &= \frac{t}{s}\left[\frac{\mathrm{Re}(u)^2}{\left(\frac{1}{\sqrt{s/t}} + t\sqrt{s/t}\right)^2} + \frac{\mathrm{Im}(u)^2}{\left(\frac{1}{\sqrt{s/t}} - t\sqrt{s/t}\right)^2} - 1\right] \end{aligned}$$

and then noticing that g_t sends $\{z \in \mathbb{C} : |z| = r\}$ to $\{u \in \mathbb{C} : \frac{\mathrm{Re}(u)^2}{(\frac{1}{r}+tr)^2} + \frac{\mathrm{Im}(u)^2}{(\frac{1}{r}-tr)^2} = 1\}$. Moreover, we can find

$$\begin{aligned} F_u(t|z|^2) &= |z|^2 \left(\frac{\mathrm{Re}\,(u)^2}{1+t|z|^2} + \frac{\mathrm{Im}\,(u)^2}{1-t|z|^2} \right) - \log(t|z|^2) + \log t \\ &= 1 + t|z|^4 \left(\frac{\mathrm{Re}\,(u)^2}{(1+t|z|^2)^2} - \frac{\mathrm{Im}\,(u)^2}{(1-t|z|^2)^2} \right) - \log(|z|^2) \\ &= 1 + t\mathrm{Re}(z^2) - \log(|z|^2), \end{aligned}$$

where for the second equality we have used that $F_u'(t|z|^2) = 0$ to simplify the calculation and for the third equality we have used that $u = z^{-1} + tz$ allows us to relate the real parts $(t|z|^2+1)\mathrm{Re}(z) = |z|^2\mathrm{Re}(u)$ and the imaginary parts $(t|z|^2-1)\mathrm{Im}(z) = |z|^2\mathrm{Im}(u)$. In our case we need to control the second moment

$$\begin{aligned} \mathbb{E}[|f_n(z)|^2] &= \frac{n!|z|^{2n}}{n^n} \mathrm{e}^{-nt\mathrm{Re}(z^2)} \sum_{k=0}^{n} \frac{t^k}{k!} \left| H_k \left(\sqrt{\frac{n}{t}} \left(\frac{1}{z} + tz \right) \right) \right|^2 \\ &= \frac{n!|z|^{2n}}{n^n} \mathrm{e}^{-nt\mathrm{Re}(z^2)} L_{n+1} \left(\sqrt{\frac{n}{n+1}} g_t(z) \right). \end{aligned}$$

By (4.2.10) and Stirling's formula, we immediately notice that

$$\frac{n!|z|^{2n}}{n^n} \mathrm{e}^{-nt\mathrm{Re}(z^2)} L_n(g_t(z)) \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{F_u''(t|z|^2)(1-t^2|z|^4)t(1-|z|^2)}}.$$

uniformly on compact sets of $\mathbb{D} \setminus \{0\}$. It is now enough to notice that the quotient $L_{n+1}(\sqrt{n/(n+1)}g_t(z))/L_n(g_t(z))$ converges uniformly on compact sets towards a nowhere zero function. To this end, we must compute the limit of $\exp((n+1)G(\sqrt{n/(n+1)}u) - nG(u))$, where $G(w) = F_w(t|g_t^{-1}(w)|^2)$. But, for u uniformly on compact sets of $\mathbb{C} \setminus \mathcal{E}_t$,

$$\mathrm{e}^{(n+1)G(\sqrt{\frac{n}{n+1}}u) - nG(u)} \xrightarrow[n\to\infty]{} \mathrm{e}^{G(u) - \frac{1}{2}\langle \nabla G(u), u \rangle}$$

so that the proof is complete with $\mathcal{F}(z) = \frac{\mathrm{e}^{G(u)-\frac{1}{2}\langle\nabla G(u),u\rangle}}{\sqrt{F_u''(t|z|^2)(1-t^2|z|^4)t(1-|z|^2)}}$, $u = g_t(z)$. □

Conclusion of the proof of Theorem 4.2.2. For every compact subset $K \subset \mathbb{D}$, Proposition 4.2.10 implies that the sequence $(||f_n||_K)_{n\geqslant 1}$ has its second moment uniformly bounded. The sequence is therefore tight from which one derives the tightness of $\{f_n\}_{n\geqslant 1}$ using Lemma 4.2.5 and Remark 4.2.6.

□

4.2.2 Convergence of the coefficients: Proof of Theorem 4.2.4

We will divide the proof in three parts. First, we show that the expected value $E[U_k^{(n)}]$ converge to zero. As a second step we show that the fluctuations are Gaussian. Finally, we identify the covariance. We will deal with traces of polynomials of A_n. Recall that

$$\mathrm{Tr}\left[A_n^k\right] = \sum_{(i_1,\ldots,i_k)\in\{1,\ldots,n\}^k} a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{k-1} i_k} a_{i_k i_1}.$$

It is convenient to think of $(i_1, \ldots, i_k)$ as a path of length k with values in $\{1, \ldots, n\}$. We use the notation $[m] = \{1, \ldots, m\}$ which will be thought of as the abelian group $\mathbb{Z}/m\mathbb{Z}$

when performing additions, and for $\psi : [k] \to [n]$ we denote $a_\psi = \prod_{i=1}^{k} a_{\psi(i)\psi(i+1)}$. Due to the invariance under permutations of the law of the entries in the matrix A_n, quantities such as $\mathbb{E}[a_\psi]$ or $\mathbb{E}[a_\psi a_\varphi]$ are invariant under $\psi \mapsto T \circ \psi$ for any permutation $T : [n] \to [n]$, i.e., $\mathbb{E}[a_\psi] = \mathbb{E}[a_{T\circ\psi}]$ or $\mathbb{E}[a_\psi a_\varphi] = \mathbb{E}[a_{T\circ\psi} a_{T\circ\varphi}]$. In the following, we may identify ψ and $T \circ \psi$ or (ψ, φ) and $(T \circ \psi, T \circ \varphi)$ for some purposes.

One can describe these equivalence classes by considering the partition induced by ψ or by (ψ, φ) as follows. Let $k_1, \ldots, k_\ell$ be positive integers and let us use the notation $\mathbf{k} = (k_1, \ldots, k_\ell)$. First recall that the information of ℓ maps $(\psi_j : [k_j] \to [n])_{1 \leqslant j \leqslant \ell}$ is contained in the map $\psi = \sqcup \psi_j : \sqcup_{j=1}^{\ell} [k_j] \to [n]$. For simplicity, we denote $[\mathbf{k}] = \sqcup_{j=1}^{\ell} [k_j]$.

Definition 4.2.11 (Path pattern)**.** Let $\psi : [\mathbf{k}] \to [n]$. The *path pattern* traced by ψ, or the *partition* induced by ψ is the partition of $[k]$ given by

$$\pi_\psi := \left\{ \psi^{-1}(x) : x \in \psi([\mathbf{k}]) \right\}.$$

We denote by $\mathcal{P}(\mathbf{k})$ the set of path patterns, or partitions, of $[\mathbf{k}]$. We use these path patterns to write, for instance in the case of a single trace that is, $\ell = 1$,

$$\operatorname{Tr}\left[A_n^k\right] = \sum_{\pi \in \mathcal{P}(k)} \mathcal{A}_\pi^{(n)}, \qquad \text{where } \mathcal{A}_\pi^{(n)} := \sum_{\substack{\psi:[k]\to[n] \\ \pi_\psi = \pi}} a_\psi. \tag{4.2.11}$$

Since the set $\mathcal{P}(\mathbf{k})$ of possible path patterns is finite, it will be convenient to study the limit of $\mathcal{A}_\pi^{(n)}$ for a given $\pi \in \mathcal{P}(\mathbf{k})$. We chose to consider path patterns because they are convenient to count. We proceed to define the directed multigraph G_ψ associated to a map $\psi = \sqcup_{j=1}^{\ell} \psi_j : [\mathbf{k}] \to [n]$ in a more explicit way in Definition 4.2.12. The motivation is that $\mathbb{E}[a_\psi]$ depends only on isomorphism class of the graph G_ψ.

Definition 4.2.12 (Graph of ψ)**.** Let $\psi = \sqcup_{j=1}^{\ell} \psi_j : [\mathbf{k}] \to [n]$. The graph $G_\psi = (V, E, s, t)$ is the directed multigraph having vertex set $V = \psi([\mathbf{k}])$, edge set $E = \sqcup_{j=1}^{\ell} E_j$ with $E_j = \{(i, i+1) : i \in [k_j]\}$, source map $s = \sqcup_{j=1}^{\ell} s_j$ with $s_j : E_j \to V$ given by $s(i, i+1) = \psi_j(i)$ and target map $t = \sqcup_{j=1}^{\ell} t_j$ with $t_j : E_j \to V$ given by $t(i, i+1) = \psi_j(i+1)$.

More concisely but equivalently for our purposes, if $I_j : [k_j] \to [\mathbf{k}]$ are the canonical inclusion maps, we may say that G_ψ is the directed multigraph with vertex set $V = \psi([\mathbf{k}])$ and edge multiset $\{(I_j(i), I_j(i+1)) : j \in [\ell], i \in [k_j]\}$. For a directed multigraph $G = (V, E, s, t)$, denote by $\overline{G} = (V, \overline{E})$ the undirected simple graph with edge set $\overline{E} = \{\{x, y\} \subset V : \text{there exists } e \in E \text{ with } \{s(e), t(e)\} = \{x, y\}\}$. So, $\overline{G}_\psi$ is the undirected simple graph (with possible loops) induced by ψ.

Notice that the isomorphism class of G_ψ depends only on the path pattern $\pi \in \mathcal{P}(\mathbf{k})$ induced by ψ. We denote by G_π this isomorphism class. When dealing with G_π for purposes where the quantities only depend on the path pattern, we may write V_π and E_π for its set of vertices and edges viewed as V_ψ and E_ψ for some ψ inducing the partition π. The same goes for $\overline{G}_\pi$. Moreover, if $\psi = \sqcup_{i=1}^{\ell} \psi_j$ and $\pi = \pi_\psi$, we can consider π_j, the path pattern induced by ψ_j or, what is the same, the partition induced by π via the canonical inclusion $I_j : [k_j] \to [\mathbf{k}]$. It will then be convenient to see G_{π_i} as a subgraph of G_π and $\overline{G}_{\pi_i}$

as a subgraph of $\overline{G}_\pi$. This suggest to consider the equivalence class of $(G_\psi, G_{\psi_1}, \ldots, G_{\psi_\ell})$ and viewing G_{π_i} as a subgraph of G_π by taking representatives of this equivalence class.

We will see below in the proof of Lemma 4.2.15 and in Proposition 4.2.17 the following behaviors for $n^{-k/2}\mathcal{A}_\pi^{(n)}$. The first two cases correspond to the random part in the limit whereas the last two cases give deterministic contributions.

1. If $\overline{G}_\pi$ is unicyclic, i.e., if it contains one and only one cycle (that may be a loop), with each edge of the cycle being traversed just once and every edge outside the cycle being traversed once in each direction in G_π, then $n^{-k/2}\mathcal{A}_\pi^{(n)}$ converges to a centered Gaussian.
2. If $\overline{G}_\pi$ is a tree where each edge is traversed twice in G_π, once in each direction (π is called a rooted plane tree in this case, see Definition 4.2.16), then $n^{-k/2}\mathcal{A}_\pi^{(n)} - n$ converges to a non-centered Gaussian. This case can be thought of as a unicyclic with a cycle of length two.
3. If $\overline{G}_\pi$ is a tree but there is an edge traversed four times in G_π, two in each direction, then $n^{-k/2}\mathcal{A}_\pi^{(n)}$ converges to a constant. Those π can be thought of as rooted plane trees with two vertices at distance two identified.
4. If $\overline{G}_\pi$ is unicyclic but with each edge traversed two times, once in each direction, then $n^{-k/2}\mathcal{A}_\pi^{(n)}$ also converges to a constant. Those π can be thought of as rooted plane trees with two vertices at distance different from two identified.
5. All other cases converge to zero.

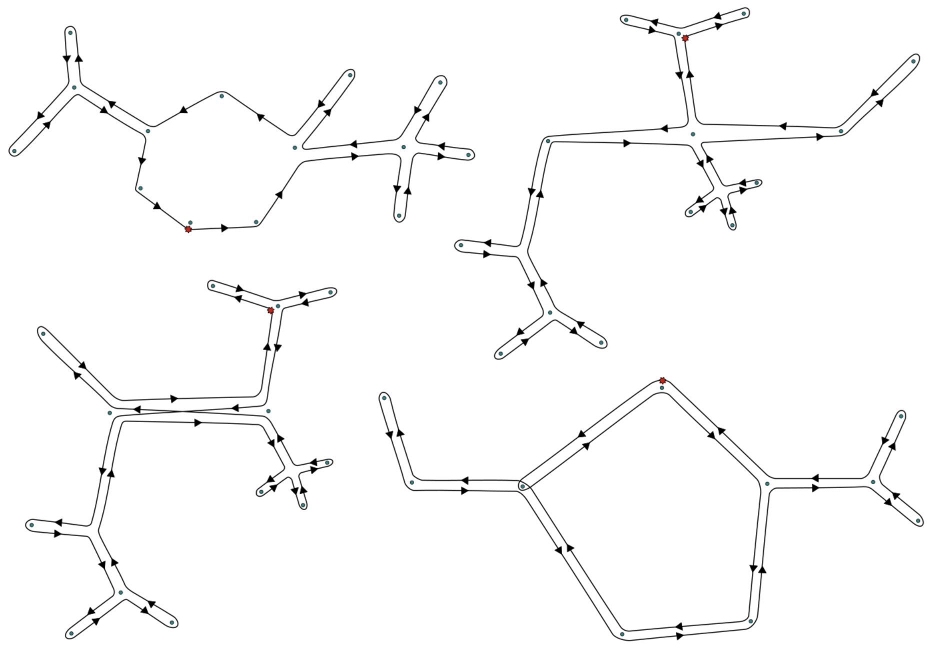

Figure 4.2: Graphs for which $n^{-k/2}\mathcal{A}_\pi^{(n)}$ has a non–trivial limit. Top left, top right, bottom left, bottom right correspond respectively to cases 1, 2, 3 and 4 above.

Before turning to the convergence of the expectation and the fluctuations of traces, we state a general lemma which will be used frequently.

Lemma 4.2.13 (Even multiplicities in trees). *Let $\psi : [k] \to [n]$ be such that $\overline{G}_\psi$ is a tree. Then, each edge of $\overline{G}_\psi$ has an even multiplicity in G_ψ with an equal number of oriented edges in each direction.*

Proof. Let $e = (i, i+1) \in E_\psi$ be an edge. Denote $u = \psi(i)$ and $v = \psi(i+1)$ its endpoints. Since ψ traces a cycle, there exists a sequence of edges $(i+1, i+2), \ldots, (i+r, i+r+1)$ such that $s((i+1, i+2)) = v$ and $t((i+r, i+r+1)) = u$ for some $1 \leqslant r \leqslant k-1$. Take such r minimal. Since $\overline{G}_\psi$ is a tree, one must have $s((i+r, i+r+1)) = v$ otherwise the graph $\overline{G}_\psi$ would have a cycle. Removing the edges $(i, i+1), \ldots, (i+r, i+r+1)$ from G_ψ shows the result by induction on $|\{e \in E_\psi : \{s(e), t(e)\} = \{u, v\}\}|$. ☐

Convergence of expected value

Recall that

$$U_k^{(n)} = \mathrm{Tr}\left[P_k\left(\frac{A_n}{\sqrt{n}}\right)\right] + nt\delta_{k=2}.$$

as defined previously in (4.2.5). The main result of this section is Proposition 4.2.14 which shows the convergence of $E\left[U_k^{(n)}\right]$ for each $k \geqslant 1$.

Proposition 4.2.14 (Vanishing Chebyshev expectation). *For every $k \geqslant 1$,*

$$\lim_{n\to\infty} \mathbb{E}\left[U_k^{(n)}\right] = 0.$$

The proof is based on the convergence of traces of monomials which is Lemma 4.2.15. As for Wigner matrices, Catalan numbers $C_p = \frac{1}{p+1}\binom{2p}{p}$ are involved.

Lemma 4.2.15 (Monomial expectation). *Let $p \geqslant 1$. Then,*

(i) $\mathbb{E}\left[\mathrm{Tr}\left[\left(\frac{A_n}{\sqrt{n}}\right)^{2p}\right]\right] - nC_pt^p \xrightarrow[n\to\infty]{} 0$ *and*

(ii) $\mathbb{E}\left[\mathrm{Tr}\left[\left(\frac{A_n}{\sqrt{n}}\right)^{2p+1}\right]\right] \xrightarrow[n\to\infty]{} 0.$

We now prove Proposition 4.2.14, using the asymptotics of Lemma 4.2.15.

Proof of Proposition 4.2.14. We begin by noticing that, if σ_t denotes the uniform probability measure on the ellipse $\mathcal{E}_t$ defined in (4.1.5),

$$\int_{\mathcal{E}_t} w^{2p}\mathrm{d}\sigma_t(w) = \frac{1}{p+1}\binom{2p}{p}t^p = C_pt^p$$

which can be obtained by using elliptic coordinates. Then, from Lemma 4.2.15, (i), the asymptotic

$$\mathbb{E}\left[\mathrm{Tr}\left[\left(\frac{A_n}{\sqrt{n}}\right)^{2p}\right]\right] = nC_pt^p + o(1) = n\int_{\mathcal{E}_t} w^{2p}\mathrm{d}\sigma_t(w) + o(1)$$

implies that

$$\mathbb{E}\left[\mathrm{Tr}\left[P_k^{(t)}\left(\frac{A_n}{\sqrt{n}}\right)\right]\right] = n\int_{\mathcal{E}_t} P_k^{(t)}(w)\mathrm{d}\sigma_t(w) + o(1).$$

It remains to compute $\int_{\mathcal{E}_t} P_k^{(t)}(w)\mathrm{d}\sigma_t(w)$. To achieve this, we define

$$\mathcal{M}(z) = \int_{\mathcal{E}_t} \log\left(1 + tz^2 - zw\right)\mathrm{d}\sigma_t(w)$$

which is holomorphic for $z \in \mathbb{D}$ with

$$\frac{d^k}{dz^k}\mathcal{M}(z) = \int_{\mathcal{E}_t} \frac{d^k}{dz^k} \log\left(1 + tz^2 - zw\right) \mathrm{d}\sigma_t(w).$$

In particular, the coefficient z^k of $\mathcal{M}(z)$ is given by

$$[z^k]\mathcal{M}(z) = -\int_{\mathcal{E}_t} \frac{P_k^{(t)}(w)}{k} \mathrm{d}\sigma_t(w).$$

To make a connection with classical Hermitian random matrix theory we can compute

$$S_t(z) = \int_{\mathcal{E}_t} \frac{\mathrm{d}\sigma_t(w)}{z-w} = \frac{1}{z}\sum_{p=0}^{\infty}\left(\frac{\sqrt{t}}{z}\right)^{2p} C_p = \frac{1}{\sqrt{t}} h\left(\frac{z}{\sqrt{t}}\right),$$

where $h(z) = \frac{z-\sqrt{z^2-4}}{2}$ is the holomorphic solution to $z = h + \frac{1}{h}$ that goes to zero at infinity, which we may recognize as the Cauchy-Stieltjes transform of the semi-circle distribution (case $t = 1$). In particular, $S_t(z^{-1} + tz) = z$ and we may connect the function S_t with $\mathcal{M}$ by taking its derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{M}(z) &= \frac{1}{z} + \left(-\frac{1}{z^2} + t\right)\int_{\mathcal{E}_t} \frac{1}{\frac{1}{z} + tz - w}\mathrm{d}\sigma_t(w) = \frac{1}{z} + \left(-\frac{1}{z^2} + t\right) S_t\left(\frac{1}{z} + tz\right) \\ &= \frac{1}{z} + \left(-\frac{1}{z^2} + t\right) z = tz.\end{aligned}$$

So, since we also know that $\mathcal{M}(0) = 0$, we obtain $\mathcal{M}(z) = \frac{tz^2}{2}$, which implies that

$$\int_{\mathcal{E}_t} P_k^{(t)}(w)\mathrm{d}\sigma_t(w) = 0 \text{ for } k \neq 2 \quad \text{while} \int_{\mathcal{E}_t} P_2^{(t)}(w)\mathrm{d}\sigma_t(w) = -t.$$

□

We now turn to the proof of Lemma 4.2.15.

Proof of Lemma 4.2.15. For $k \geqslant 1$, write

$$\mathbb{E}\left[\mathrm{Tr}\left[\left(\frac{A_n}{\sqrt{n}}\right)^k\right]\right] = n^{-k/2}\sum_{\pi\in\mathcal{P}(k)} \mathbb{E}\left[\mathcal{A}_\pi^{(n)}\right] = n^{-k/2}\sum_{\pi\in\mathcal{P}(k)} C_\pi^{(n)}\alpha_\pi,$$

with α_π being the common value of $\mathbb{E}[a_\psi]$ for ψ inducing the partition π and $C_\pi^{(n)}$ is the number of $\psi : [k] \to [n]$ that induce π. Since the choice of ψ inducing π amounts to choosing the image of each block of π, we find that, for $n \geqslant k$, $C_\pi^{(n)} = n^{\underline{|\pi|}}$, where $|\pi|$ denotes the number of blocks of π and $n^{\underline{k}} = n(n-1)\dots(n-k+1)$ is the falling factorial. Let us investigate the cases where $\alpha_\pi \neq 0$.

As entries are centered, for α_π not to vanish, every edge of G_π has to be at least double. Let us denote this set by $\mathcal{D}_k = \{\pi \in \mathcal{P}(k) : G_\pi \text{ has no simple edges}\}$. Since for $\pi \in \mathcal{D}_k$ we have $2|\overline{E}_\pi| \leqslant |E_\pi|$, we obtain $|V_\pi| \leqslant |\overline{E}_\pi| + 1 \leqslant \frac{k}{2} + 1$ so that, for $n \geqslant k$,

$$\mathbb{E}\left[\mathrm{Tr}\left[\left(\frac{A_n}{\sqrt{n}}\right)^k\right]\right] = n^{-k/2}\sum_{\ell=1}^{\lfloor\frac{k}{2}+1\rfloor} n^{\underline{\ell}} \sum_{\pi\in\mathcal{D}_k, |\pi|=\ell} \alpha_\pi.$$

For odd values of k, the only possible contribution to the limit comes from $\ell = (k+1)/2$. For this ℓ, we would have $|V_\pi| = (k+1)/2$ and, since $|V_\pi| - 1 \leqslant |\overline{E}_\pi| \leqslant k/2$, we must have $|\overline{E}_\pi| = (k-1)/2$ so that $\overline{G}_\pi$ is a tree. The fact that $|\overline{E}_\pi| = (k-1)/2$ and that each edge is at least double in E_π tells us that there is exactly one edge that is triple in E_π. This would contradict the result of Lemma 4.2.13 so that there is no $\pi \in \mathcal{D}_k$ satisfying $|\pi| = (k+1)/2$. Then, for odd k the expected value goes to zero.

For even values $k = 2p$,

$$\mathbb{E}\left[\mathrm{Tr}\left[\left(\frac{A_n}{\sqrt{n}}\right)^{2p}\right]\right] = n \sum_{\pi \in \mathcal{D}_{2p}, |\pi| = p+1} \alpha_\pi + \left[-\binom{p+1}{2} \sum_{\pi \in \mathcal{D}_{2p}, |\pi| = p+1} \alpha_\pi + \sum_{\pi \in \mathcal{D}_{2p}, |\pi| = p} \alpha_\pi\right] + O\Big(\frac{1}{n}\Big).$$

For $\pi \in \mathcal{D}_{2p}$ satisfying $|\pi| = p+1$, the graph $\overline{G}_\pi$ has p edges and $p+1$ vertices so that it is a tree and π draws a path on $\overline{G}_\pi$ with exactly two edges passing through each edge in $\overline{E}_\pi$, necessarily once in each direction according to Lemma 4.2.13. Denote by $\mathcal{T}_p$ the set of these path patterns, also known as rooted plane trees. Then, $\alpha_\pi = t^p$ for every $\pi \in \mathcal{T}_p$ so that

$$\sum_{\pi \in \mathcal{D}_{2p}, |\pi| = p+1} \alpha_\pi = |\mathcal{T}_p| t^p.$$

For $\pi \in \mathcal{D}_{2p}$ satisfying $|\pi| = p$, we can either have $|\overline{E}_\pi| = p$ or $p-1$.

- If $|\overline{E}_\pi| = p$, then $\overline{G}_\pi$ is a graph with p edges and p vertices so that it is a unicyclic graph. For α_π to be non-zero, the path drawn by π on $\overline{G}_\pi$ has to have exactly two edges passing in opposite directions through each edge in $\overline{E}_\pi$ (its cycle can be a loop). The value of α_π in this case is $\mathbb{E}[a_{12}a_{21}]^p = t^p$ in the case of no loops and $\mathbb{E}[a_{12}a_{21}]^{p-1}\mathbb{E}[a_{11}^2] = t^{p-1}t$ in the case where there is a loop which turns out to be the same number. We denote by $\mathcal{N}_p$ the set of all these path patterns. This case corresponds to the graph located in the bottom–right corner of Figure 4.2.

- If $|\overline{E}_\pi| = p-1$, then $\overline{G}_\pi$ is a graph with $p-1$ edges and p vertices so that it is a tree. Since a closed path drawn on a tree should pass an even number of times by each edge with the same number of times in each direction by Lemma 4.2.13, every edge of $\overline{G}_\pi$ should be double in G_π except only for one edge of multiplicity four. The value of α_π in this case is $\mathbb{E}[(a_{12}a_{21})^2]\mathbb{E}[a_{12}a_{21}]^{p-2} = (2t^2)t^{p-2} = 2t^p$ and we denote by $\mathcal{N}'_p$ the set of these path patterns. This case corresponds to the graph located in the bottom–left corner of Figure 4.2.

We will now show that the $O(1)$ term in the asymptotic development of traces vanishes by showing the combinatorial equality

$$\binom{p+1}{2}|\mathcal{T}_p| = |\mathcal{N}_p| + 2|\mathcal{N}'_p|. \tag{4.2.12}$$

The idea behind (4.2.12) is that, given an element of $\mathcal{T}_p$, one can choose two vertices and identify them to obtain an element either of $\mathcal{N}_p$ if the vertices are not at distance two or of $\mathcal{N}'_p$ if the chosen vertices are at distance two. More precisely, consider

$$\Phi : \bigsqcup_{\pi \in \mathcal{T}_p} \{\{a,b\} \subset \pi : a \neq b\} \to \mathcal{N}_p \cup \mathcal{N}'_p$$

defined by taking $\pi \in \mathcal{T}_p$ and a pair of blocs a_1, a_2 of $\pi = \{a_1, a_2, a_3, \ldots, a_{p+1}\}$ to the partition $\{a_1 \cup a_2, a_3, \ldots, a_{p+1}\}$. We may notice that an element of $\mathcal{N}_p$ has a unique preimage by Φ. Here is a possible way to see this. Let $\pi \in \mathcal{N}_p$ and let $\psi : [k] \to [n]$ be a function which induces π. Consider the smallest index $\ell \in [k]$ such that:

- edges $(\ell, \ell+1)$ and $(\ell+r, \ell+r+1)$ for some $r \geqslant 1$ are in opposite directions and belong to the cycle of $\overline{G}_\pi$,
- edges $(\ell+1, \ell+2), \dots, (\ell+r-1, \ell+r)$ belong to a tree component in $\overline{G}_\pi$.

Then, as $\overline{G}_\pi$ has a cycle, there exists $s \in [k]$ such that $\psi(s) \neq \psi(\ell)$ and $\psi(s+1) = \psi(\ell+1)$. In π, there is the block

$$\{t_1, \dots, t_p, \ell+1, \ell+r, s+1, t_{p+1}, \dots, t_q\}$$

for some $t_1, \dots, t_q \in [k]$ and $q \geqslant 0$. The preimage of π is given by splitting the above block into two distinct blocks:

$$\{t_1, \dots, t_p, \ell+1, \ell+r\} \sqcup \{s+1, t_{p+1}, \dots, t_q\}.$$

On the other hand, an element of $\mathcal{N}'_p$ has exactly two preimages by Φ each one being obtained by splitting blocks one of the two endpoints of the "quadruple" edge (see Figure 4.3). Then,

$$\binom{p+1}{2}|\mathcal{T}_p| = |\mathcal{N}_p| + 2|\mathcal{N}'_p|$$

so that

$$\mathbb{E}\left[\mathrm{Tr}\left[\left(\frac{A_n}{\sqrt{n}}\right)^{2p}\right]\right] = n|\mathcal{T}_p|t^p - \left[\binom{p+1}{2}|\mathcal{T}_p|t^p - |\mathcal{N}_p|t^p - |\mathcal{N}'_p|2t^p\right] + O\Big(\frac{1}{n}\Big) = nC_pt^p + O\Big(\frac{1}{n}\Big).$$

$\square$

Convergence of fluctuations

Recall the definition $\mathcal{A}_\pi^{(n)} = \sum_{\substack{\psi:[k]\to[n] \\ \pi_\psi = \pi}} a_\psi$ from (4.2.11) and let us use the following notation.

Definition 4.2.16 (Rooted plane tree and unicyclic graph)**.** Let $k \geqslant 1$ and $\pi \in \mathcal{P}(k)$.

- We say that $\pi \in \mathcal{P}(k)$ is a *rooted plane tree* if $\overline{G}_\pi$ is a tree and each edge in G_π is double, once in each direction. This case corresponds to the graph located in the top–right corner of Figure 4.2.
- We say that $\pi \in \mathcal{P}(k)$ is a *rooted plane unicyclic graph* if $\overline{G}_\pi$ has a unique cycle, with each edge in G_π being simple if it belongs to the unique cycle and double, once in each direction, if it does not belong to the unique cycle. This case corresponds to the graph located in the top–left corner of Figure 4.2.

Proposition 4.2.17 (Gaussian limit process)**.** *The family*

$$\left(n^{-k/2}\mathcal{A}_\pi^{(n)} - n^{-k/2}\mathbb{E}\left[\mathcal{A}_\pi^{(n)}\right]\right)_{\pi\in\mathcal{P}(k), k\geqslant 1}$$

converges to a Gaussian process as n goes to infinity. Moreover, the only π that give non-trivial limits are the rooted plane trees and the rooted plane unicyclic graphs, i.e., if π is neither of those, the sequence $n^{-k/2}\mathcal{A}_\pi^{(n)} - n^{-k/2}\mathbb{E}\left[\mathcal{A}_\pi^{(n)}\right]$ converges to zero in law.

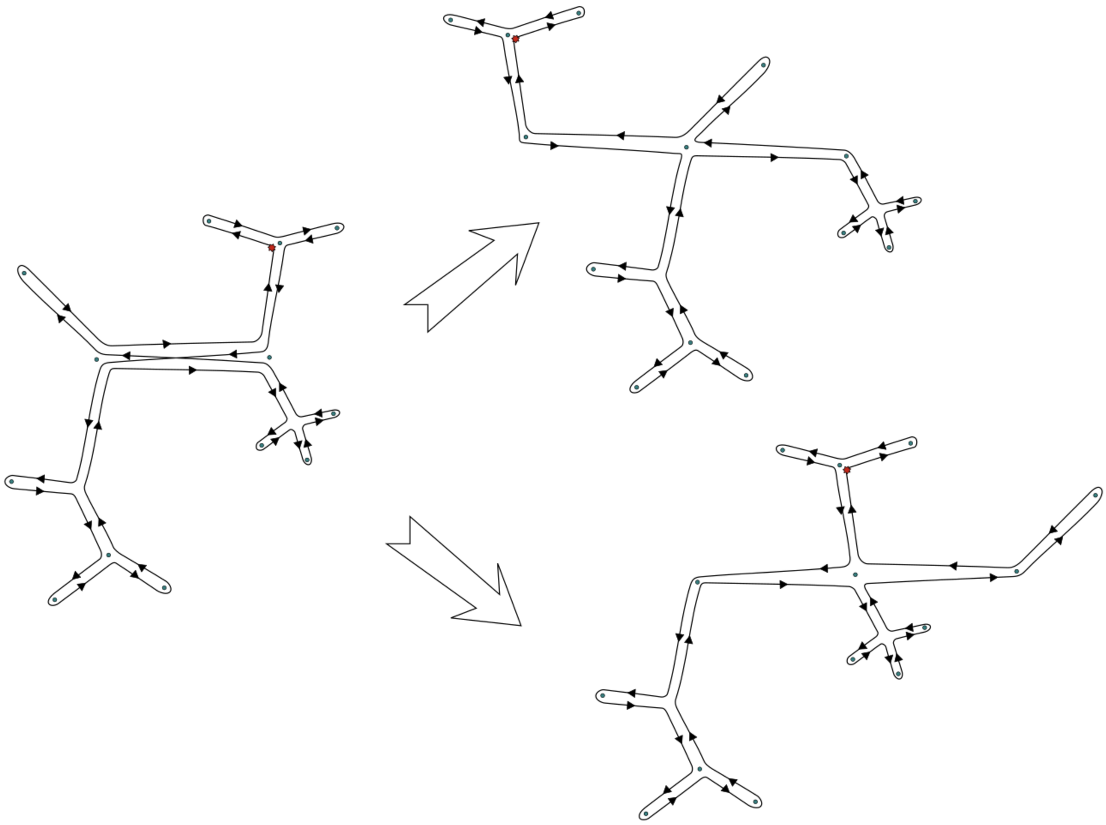

Figure 4.3: A tree with a quadruple edge with the two possible ways of ungluing it to obtain a planar tree.

Proof. Let $k_1, \dots, k_\ell \geqslant 1$ and denote $k = k_1 + \cdots + k_\ell$. For each $i \in \{1, \dots, \ell\}$ choose $\pi_i \in \mathcal{P}(k_i)$ and exponents $s_i \in \{\cdot, *\}$ with the conventions $x^{(\cdot)} = x$ and $x^{(*)} = \overline{x}$. Our goal is to show that

$$\lim_{n\to\infty} n^{-k/2}\mathbb{E}\Big[\prod_{i=1}^{\ell}(\mathcal{A}_{\pi_i} - \mathbb{E}[\mathcal{A}_{\pi_i}])^{(s_i)}\Big] = \mathbb{E}\Big[\prod_{i=1}^{\ell} Y_i^{(s_i)}\Big]$$

where $(Y_i)_{1\leqslant i\leqslant \ell}$ is a Gaussian vector on $\mathbb{C}^\ell$. Let us write

$$n^{-k/2}\mathbb{E}\Big[\prod_{i=1}^{\ell}(\mathcal{A}_{\pi_i} - \mathbb{E}[\mathcal{A}_{\pi_i}])^{(s_i)}\Big] = n^{-k/2} \sum_{\substack{\psi:[\mathbf{k}]\to[n] \\ \pi_{\psi_j}=\pi_j}} \mathbb{E}\Big[\prod_{j=1}^{\ell}(a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}\Big],$$

where we recall that $[\mathbf{k}] = \sqcup_{j=1}^{\ell}[k_j]$ and $\psi = \sqcup_{j=1}^{\ell}\psi_j$ with $\psi_j : [k_j] \to [n]$.
Since $\mathbb{E}[\prod_{j=1}^{\ell}(a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}]$ depends only on the partition induced by ψ, we can write

$$\sum_{\substack{\psi:[\mathbf{k}]\to[n] \\ \pi_{\psi_j}=\pi_j}} \mathbb{E}\Big[\prod_{j=1}^{\ell}(a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}\Big] = \sum_{\substack{\tau\in\mathcal{P}(\mathbf{k}) \\ \tau_j=\pi_j}} C_\tau^{(n)}\beta_\tau$$

with the following notation. The partition $\tau_j \in \mathcal{P}(k_j)$ is the one induced by τ via the canonical inclusion $I_i : [k_j] \to [\mathbf{k}]$, i.e., the part of the path pattern traced by the j-th path, $\beta_\tau = \mathbb{E}[\prod_{j=1}^{\ell}(a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}]$ for any ψ inducing τ and $C_\tau^{(n)}$ is the cardinal of $\{\psi : [\mathbf{k}] \to [n] : \psi \text{ induces } \tau\}$ which equals $n^{\underline{|\tau|}} = n(n-1)\dots(n-|\tau|+1)$ if $n \geqslant k$.

The question now reduces to understand that $n^{-k/2}C_\tau^{(n)}$ has a limit if $\beta_\tau \neq 0$ and to understand for which of those τ the limit is not zero. This is the purpose of Lemmas 4.2.18 and 4.2.19 that we state and prove now. The end of the proof of Proposition 4.2.17 will follow afterwards.

Lemma 4.2.18. *If $\beta_\tau \neq 0$ then each edge of $\overline{G}_\tau$ is at least double in G_τ and for each $i \in \{1,\dots,\ell\}$ there is $j \in \{1,\dots,\ell\}$ different from i such that $\overline{G}_{\tau_i}$ and $\overline{G}_{\tau_j}$ share an edge.*

Proof. Let $\psi : [\mathbf{k}] \to [n]$ which induces τ. If G_ψ has a simple edge then there is a unique $i \in \{1,\dots,\ell\}$ such that the edge $(r, r+1) \in E_i$ of G_{ψ_i} for some $r \in [k_i]$ is simple. In that case $\mathbb{E}[a_{\psi_i}] = 0$ since G_{ψ_i} has a simple edge and since $a_{\psi_i(r),\psi_i(r+1)}$ is centered,

$$\beta_\tau = \mathbb{E}\Bigg[\prod_{j=1}^{\ell}(a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}\Bigg] = \mathbb{E}\left[a_{\psi_i(r),\psi_i(r+1)}^{(s_i)}\right]\mathbb{E}\Bigg[\prod_{j\neq i}(a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}\Bigg] = 0.$$

Let $i \in \{1,\dots,\ell\}$. If there is no index $j \in \{1,\dots,\ell\}$ different from i such that $\overline{G}_{\tau_i}$ and $\overline{G}_{\tau_j}$ share an edge, the random variable $(a_{\psi_i} - \mathbb{E}[a_{\psi_i}])^{(s_i)}$ would be independent of the product $\prod_{j\neq i}(a_{\psi_j} - \mathbb{E}[a_{\psi_j}])^{(s_j)}$ so that, since $(a_{\psi_i} - \mathbb{E}[a_{\psi_i}])^{(s_i)}$ is centered, β_τ would be zero. $\square$

Recall that $k = k_1 + \dots + k_\ell$.

Lemma 4.2.19. *Suppose that $\beta_\tau \neq 0$. Then, $|\tau| \leqslant k/2$. Moreover, if some connected component of $\overline{G}_\tau$ involves three or more $\overline{G}_{\tau_i}$, then the strict inequality $|\tau| < k/2$ holds.*

Proof. We will actually show this lemma under the conclusions of Lemma 4.2.18.

Recall that $|\tau|$ counts the number of vertices in G_τ so that we want to show that $|V_\tau| \leqslant k/2$. A connected component of $\overline{G}_\tau$ is formed by some $G_{\tau_{i_1}}, \dots, G_{\tau_{i_s}}$ so that it is enough to prove this inequality for τ restricted to $[k_{i_1}] \sqcup \dots \sqcup [k_{i_s}]$. In other words, we may assume without loss of generality that $\overline{G}_\tau$ is connected.

Since each edge is at least double we have that $|\overline{E}_\tau| \leqslant k/2$. The inequality $|V_\tau| \leqslant |\overline{E}_\tau| + 1$ tells us that $|V_\tau| \leqslant k/2+1$ and we need to understand why $|V_\tau|$ cannot be in $(k/2, k/2+1]$.

- If k is even and $|V_\tau| = k/2+1$ then $|\overline{E}_\tau| = k/2$ so that $\overline{G}_\tau$ is a tree and each edge is double. Since τ_i traces a closed path in this tree, it must traverse each edge twice, once in each direction by Lemma 4.2.13. This implies that $\overline{G}_{\tau_i}$ does not share edges with any other $\overline{G}_{\tau_j}$ because each edge of G_τ is double which contradicts $\beta_\tau \neq 0$ by Lemma 4.2.18.

- If k is odd and $|V_\tau| = k/2 + 1/2$ then $|\overline{E}_\tau| = k/2 - 1/2$ so that again $\overline{G}_\tau$ is a tree. But, using Lemma 4.2.13, a closed path in a tree contains an even number of edges so that each k_i has to be even which contradicts that $k = k_1 + \dots + k_\ell$ is odd.

If $|\tau| = k/2$ then $|\overline{E}|$ can be either $k/2$ or $k/2-1$. The first case happens when $\overline{G}_\tau$ is unicyclic and the second case happens when it is a tree. We need to see why in this case ℓ must be 2, i.e., why there can be only two $\overline{G}_{\tau_i}$ forming $\overline{G}_\tau$.

- Suppose that $|\tau| = |\overline{E}| = k/2$ so that $\overline{G}_\tau$ is unicyclic. Then, each τ_i has to traverse each edge of the unique cycle at least once because, if not, τ_i would draw a path on a tree ($\overline{G}_\tau$ with that edge removed) so that each edge would be double and, since each edge of G_τ is precisely double, G_{τ_i} would not share edges with the other G_{τ_j}. But there cannot be three G_{τ_i} passing through the unique cycle because each edge is double so that there has to be exactly two G_{τ_i}. In particular, G_{τ_i} is also unicyclic and G_τ is made of G_{τ_i} and G_{τ_j} by gluing them along the unique cycle because, if an edge is traversed at least once by τ_i then it is traversed exactly twice, once in each direction (if not, we would be able to form a cycle passing through this edge).

- Suppose that $|\tau| = |\overline{E}| + 1 = k/2$ so that $\overline{G}_\tau$ is a tree. In this case, this implies in particular, that $\overline{G}_{\tau_i}$ is a tree with each of its edges traversed twice by τ_i, once in each direction. Take τ_i and τ_j for $i \neq j$ such that G_{τ_i} and G_{τ_j} share an edge. This implies that this shared edge is at least "quadruple". Since $|\overline{E}| = k/2 - 1$, $|E| = k/2$ and each edge is at least double we must have that every edge of G_τ is precisely double except for the "quadruple edge. This implies that any other $\overline{G}_{\tau_p}$ cannot share an edge with any other G_{τ_q} which cannot happen by Lemma 4.2.18. So, G_τ is made of two trees G_{τ_i} and G_{τ_j} by gluing them along an edge (which may be thought of as a degenerate cycle).

□

We turn back to the proof of Proposition 4.2.17. Lemmas 4.2.18 and 4.2.19 already show the Gaussian behavior. Indeed, let us denote by $\mathcal{P}_2(\pi_1, \ldots, \pi_\ell)$ the set of partitions $\tau \in \mathcal{P}(\mathbf{k})$ such that $\tau_i = \pi_i$ for every $i \in [\ell]$ and such that $\beta_\tau \neq 0$, i.e that satisfies the conditions of Lemmas 4.2.18 and 4.2.19. The proof of Lemma 4.2.19 showed that connected components of G_τ come from two τ_i's which are paired according to a common cycle or via an edge which will be quadruple if both are rooted plane trees. Then, by Lemma 4.2.19, there exists a pair partition that we denote Π_τ of $[\ell]$ where a block is $\{i, j\}$ if some connected component of G_τ is formed by G_{τ_i} and G_{τ_j}. In particular, ℓ has to be even for $\mathcal{P}_2(\pi_1, \ldots, \pi_\ell)$ not to be empty. In all cases,

$$\lim_{n\to\infty} n^{-k/2}\mathbb{E}\Big[\prod_{i=1}^{\ell}(\mathcal{A}_{\pi_i} - \mathbb{E}[\mathcal{A}_{\pi_i}])^{(s_i)}\Big] = \sum_{\tau\in\mathcal{P}_2(\pi_1,\ldots,\pi_\ell)} \beta_\tau$$

which would be zero if ℓ is odd. Denote by $\mathcal{P}_2(\ell)$ the set of pair partitions of $[\ell]$. For each $\tau \in \mathcal{P}_2(\pi_1, \ldots, \pi_\ell)$ consider the family of partitions $(\tau_a)_{a\in\Pi_\tau}$, where if $a = \{i, j\}$ is a block of Π_τ, the partition $\tau_a \in \mathcal{P}_2(\pi_i, \pi_j)$ is the one induced on $[k_i] \sqcup [k_j]$ by τ. We identify $\mathcal{P}_2(\pi_i, \pi_j) \simeq \mathcal{P}_2(\pi_j, \pi_i)$ so that the order is not important. The assignment $\tau \mapsto (\tau_a)_{a\in\Pi_\tau}$ defines a bijection

$$\mathcal{P}_2(\pi_1, \ldots, \pi_\ell) \to \bigsqcup_{\Pi\in\mathcal{P}_2(\ell)} \prod_{\{i,j\}\in\Pi} \mathcal{P}_2(\pi_i, \pi_j).$$

Notice that, since the components of G_τ are independent, $\beta_\tau = \prod_{a\in\Pi_\tau} \beta_{\tau_a}$ where $\beta_{\tau_a} =$

$\mathbb{E}[(\mathcal{A}_{\tau_i} - \mathbb{E}[\mathcal{A}_{\tau_i}])^{(s_i)}(\mathcal{A}_{\tau_j} - \mathbb{E}[\mathcal{A}_{\tau_j}])^{(s_j)}]$ whenever $a = \{i, j\}$. Then, we may write

$$\begin{aligned}\sum_{\tau \in \mathcal{P}_2(\pi_1,\ldots,\pi_\ell)} \beta_\tau &= \sum_{\Pi \in \mathcal{P}_2(\ell)} \sum_{\tau : \Pi_\tau = \Pi} \beta_\tau \\ &= \sum_{\Pi \in \mathcal{P}_2(\ell)} \sum_{\tau : \Pi_\tau = \Pi} \prod_{a \in \Pi} \beta_{\tau_a} \\ &= \sum_{\Pi \in \mathcal{P}_2(\ell)} \prod_{\{i,j\} \in \Pi} \left(\sum_{\tau \in \mathcal{P}_2(\pi_i,\pi_j)} \beta_\tau \right).\end{aligned}$$

where we have used the bijection described above together with the distributive property. By Isserlis–Wick's theorem, we know that

$$\mathbb{E}\left[\prod_{i=1}^{\ell} Y_i^{(s_i)}\right] = \sum_{\Pi \in \mathcal{P}_2(\ell)} \prod_{\{i,j\} \in \Pi} \sum_{\tau \in \mathcal{P}_2(\pi_i,\pi_j)} \beta_\tau \tag{4.2.13}$$

if $(Y_1, \ldots, Y_\ell)$ is a centered Gaussian vector with covariances

$$\mathbb{E}\left[Y_i^{(s_i)} Y_j^{(s_j)}\right] = \sum_{\tau \in \mathcal{P}(\pi_i,\pi_j)} \beta_\tau.$$

Since the right-hand side of (4.2.13) is the limit of covariances, we have shown that $(n^{-k/2}\mathcal{A}_\pi^{(n)} - n^{-k/2}\mathbb{E}[\mathcal{A}_\pi^{(n)}])_{\pi \in \mathcal{P}(k), k \geqslant 1}$ converges to a Gaussian family. In fact, in the proof of Lemma 4.2.19 we already found what graphs will contribute and how.

Lemma 4.2.20 (Contributing graphs)**.** *If $\mathcal{P}_2(\pi_i, \pi_j)$ is non-empty then both π_i and π_j are either plane rooted trees or they are plane rooted unicyclic graphs whose cycles have the same length.*

Proof. The proof is contained in the proof of Lemma 4.2.19. □

This concludes the proof of Proposition 4.2.17. Figure 4.4 gives an example of contributions described by Lemma 4.2.20.

□

Remark 4.2.21. A possible way to prove Theorem 4.2.4 would be to compute explicitly the limiting Gaussian family corresponding to traces of powers. These Gaussians variables would not be independent. One would have to show that their covariance is diagonalised by the Chebyshev polynomials thereby implying that the variables $\left(U_k^{(n)}\right)_{k \geqslant 1}$ are independent. However, in Section 4.2.2, we chose a different approach which provides more intuition as to why the Chebyshev polynomials are the right family for diagonalizing the covariance.

Variance identification

Proof of Theorem 4.2.4. Let us denote by $\mathcal{C}_k^{(n)}$ the set of $\psi : [k] \to [n]$ such that the associated graph π_ψ is a rooted plane unicyclic graph or a rooted plane tree. For such ψ, let denote $c(\psi)$ the set of edges in E_ψ that are single, i.e, which form the cycle of $\overline{G}_\psi$. Note that this set is empty if π_ψ is a rooted plane tree. Likewise, denote by $\ell(\psi)$ the set of edges in E_ψ incident to a vertex of degree one in $\overline{G}_\psi$, that is, edges which are leaves in trees anchored on the cycle of $\overline{G}_\psi$. Recall that

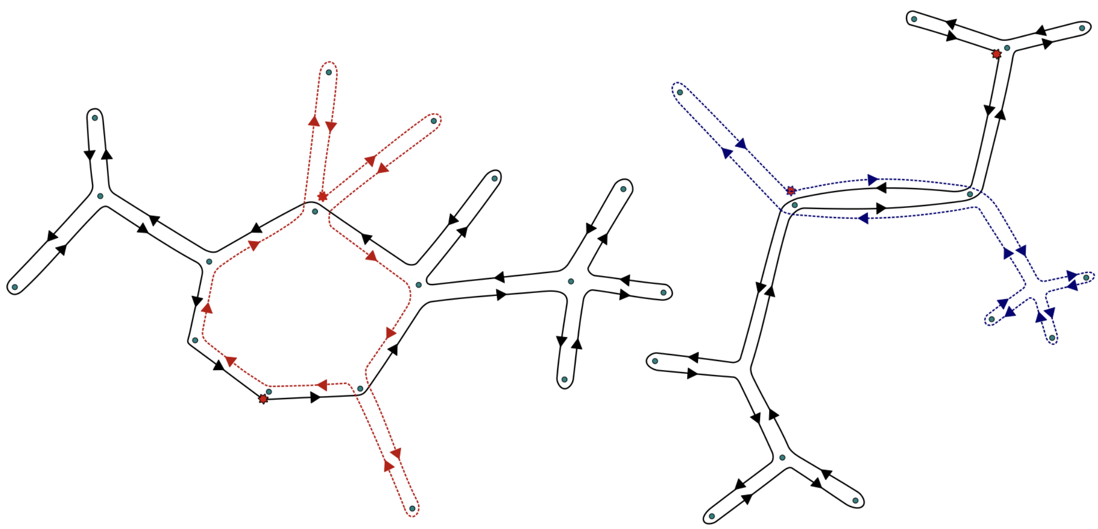

Figure 4.4: Two elements of $\mathcal{P}_2(\pi_1, \pi_2)$ where π_1, π_2 are both rooted unicyclic graphs (left) or both plane rooted trees (right).

$$\left(n^{-k/2}\mathrm{Tr}\left[A_n^k\right] - n^{-k/2}\mathbb{E}\left[\mathrm{Tr}\left[A_n^k\right]\right]\right) - n^{-k/2}\sum_{\psi\in\mathcal{C}_k^{(n)}}(a_\psi - \mathbb{E}[a_\psi]) \xrightarrow[n\to\infty]{\text{law}} 0.$$

The idea is to regularize these terms and to consider instead, for any $\psi \in \mathcal{C}_k^{(n)}$,

$$\hat{a}_\psi := \prod_{e\in c(\psi)} a_e \prod_{e\in\ell(\psi)} (a_e a_{e^*} - t)$$

where for an edge $e = (i, i+1)$, $a_e = a_{\psi(i),\psi(i+1)}$ and $e^* = (i+1, i)$. A nice property of this term is that it has zero expected value even if π_ψ is a rooted plane tree. Consider the sum

$$:\mathrm{Tr}(A_n^k):\; := \sum_{\psi\in\mathcal{C}_k^{(n)}} \hat{a}_\psi.$$

By the same arguments as in Section 4.2.2, we know that $(:\mathrm{Tr}(A_n^k):)_{k\geqslant 1}$ converges to a Gaussian process as $n\to\infty$. For a rooted plane unicyclic graph or a rooted plane tree π, we define

$$\hat{\mathcal{A}}_\pi^{(n)} := \sum_{\substack{\psi:[k]\to[n]\\ \pi_\psi=\pi}} \hat{a}_\psi.$$

In the same way, $(\hat{\mathcal{A}}_\pi^{(n)})_\pi$ converges to a Gaussian process indexed by rooted plane unicyclic graphs and rooted plane trees. Let us check that for any pair of rooted plane unicyclic graphs or trees π_1 and π_2,

$$\lim_{n\to\infty}\mathbb{E}\left[\hat{\mathcal{A}}_{\pi_1}^{(n)}\hat{\mathcal{A}}_{\pi_2}^{(n)}\right] = 0 \text{ and } \lim_{n\to\infty}\mathbb{E}\left[\hat{\mathcal{A}}_{\pi_1}^{(n)}\overline{\hat{\mathcal{A}}_{\pi_2}^{(n)}}\right] = 0$$

whenever one of them is not a cycle, that is, such that either $c(\psi_i) \neq E_{\psi_1}$ or $c(\psi_2) \neq E_{\psi_2}$ for any ψ_1, ψ_2 inducing π_1 and π_2 respectively. Since the leaves are centered in $\hat{\mathcal{A}}_{\pi_1}^{(n)}$ and

$\hat{\mathcal{A}}^{(n)}_{\pi_2}$, the only possible non-vanishing contributions to the variance arise when π_1 and π_2 that either both have no leaves or share the same leaves. The case where both have no leaves corresponds exactly to the desired condition mentioned above. Recall that the connected component must consist of double edges so that $\overline{G}_\pi$ has either a single cycle, or a quadruple edge in which case $\overline{G}_\pi$ is a tree. If π_1 and π_2 have a common leaf, this would result in a quadruple edge in G_π. Therefore, both G_{π_1} and G_{π_2} are plane rooted trees and the condition of having the same leaves implies that both $\overline{G}_{\pi_1}$ and $\overline{G}_{\pi_2}$ consist of only one edge. The latter case corresponds to two cycles of length two. Therefore, the variances that will not have a zero limit are those between two cycles of the same length so that, if c_k denotes the partition $\{\{1\},\dots,\{k\}\}$, we have in law,

$$\lim_{n\to\infty}\left(n^{-k/2}\,{:}\mathrm{Tr}(A_n^k){:}\;-n^{-k/2}\hat{\mathcal{A}}_{c_k}\right)=0.$$

If Y_k denotes the limit in law of $n^{-k/2}\hat{\mathcal{A}}_{c_k}$, we would have, for $k\geqslant 3$,

$$\mathbb{E}[(Y_k)^2]=k\mathbb{E}[a_{12}a_{21}]^k=kt^k \quad\text{ and }\quad \mathbb{E}[|Y_k|^2]=k\mathbb{E}[|a_{12}|^2]^k=k.$$

For $k=2$, we have $\mathbb{E}[(Y_2)^2]=2\mathbb{E}[(a_{12}a_{21}-t)^2]=2t^2$ and $\mathbb{E}[(Y_2)^2]=2\mathbb{E}[|a_{12}a_{21}-t|^2]=2$ and, for $k=1$, $\mathbb{E}[(Y_1)^2]=\mathbb{E}[a_{11}^2]=t^2$ and $\mathbb{E}[|Y_1|^2]=\mathbb{E}[|a_{11}|^2]=1$.

It now remains to express the normalized traces ${:}\mathrm{Tr}(A_n^k){:}$ in terms of the traces of Chebyshev polynomials, that is, in terms of the variables $U_k^{(n)}$. For simplicity, we forget the initial instant of ψ, i.e., we consider the equivalence relation generated by $\psi\sim\psi\circ\sigma$ with $\sigma:[k]\to[k]$ given by $\sigma(i)=i+1$ and denote by $\mathcal{D}_k^{(n)}$ the quotient of $\mathcal{C}_k^{(n)}$ by this equivalence relation. Notice that a_η and $\hat{a}_\eta$ are well-defined for $\eta\in\mathcal{D}_k^{(n)}$ and that, since each equivalence class contains k elements,

$${:}\mathrm{Tr}(A_n^k){:}=\sum_{\psi\in\mathcal{C}_k^{(n)}}\hat{a}_\psi=k\sum_{\eta\in\mathcal{D}_k^{(n)}}\hat{a}_\eta.$$

We may write the sum of $\hat{a}_\eta$ over $\eta\in\mathcal{D}_k^{(n)}$ as a weighted sum of $a_\theta-\mathbb{E}[a_\theta]$ with $\theta\in\mathcal{D}_{k-2j}^{(n)}$ for $j\geqslant 0$ since each time we erase a leave we are erasing two edges. Fix $j\geqslant 0$ and let us find the weight of $\theta\in\mathcal{D}_{k-2j}^{(n)}$. To construct an element from $\mathcal{D}_k^{(n)}$ by adding leaves to θ, we would have to add j of those leaves. We should choose j instants (with possible repetitions) among the $k-2j$ instants in the path θ to introduce these leaves. This would give us $\binom{k-2j+j-1}{j}=\binom{k-j-1}{j}$ choices. Then, we would have $(n-(k-2j))^{\underline{j}}\sim n^j$ choices for the vertices associated to the leaves. This means that the weight of $a_\theta-\mathbb{E}[a_\theta]$ is

$$\binom{k-j-1}{j}(n-(k-2j))^{\underline{j}}(-t)^j.$$

Note that the case where $k-2j=0$ is not relevant because our variables $\hat{a}_\psi$ are centered. We can write

$$\begin{aligned}n^{-k/2}\sum_{\psi\in\mathcal{C}_k^{(n)}}\hat{a}_\psi&=n^{-k/2}k\sum_{0\leqslant j\leqslant k/2}\binom{k-j-1}{j}(-t)^j(n-(k-2j))^{\underline{j}}\sum_{\theta\in\mathcal{D}_{k-2j}^{(n)}}(a_\theta-\mathbb{E}[a_\theta])\\&=\sum_{0\leqslant j\leqslant k/2}\frac{k}{k-2j}\binom{k-j-1}{j}(-t)^j\frac{(n-(k-2j))^{\underline{j}}}{n^j}\left(n^{-(k-2j)/2}\sum_{\psi\in\mathcal{C}_{k-2j}^{(n)}}(a_\psi-\mathbb{E}[a_\psi])\right).\end{aligned}$$

Notice then that whenever $k-2j \neq 0$,

$$\frac{k}{k-2j}\binom{k-j-1}{j}(-t)^j = \frac{k}{k-j}\binom{k-j}{j}(-t)^j = \alpha_{k-2j}^{(k)}$$

where $\left(\alpha_{k-2j}^{(k)}\right)_j$ are the coefficients of P_k given in (4.2.2). Since $\frac{(n-(k-2j))^{\underline{j}}}{n^j}$ converges to 1, the limit of $n^{-k/2}\sum_{\psi\in\mathcal{C}_k^{(n)}}\hat{a}_\psi$ coincides with the limit of

$$\mathrm{Tr}\Big(P_k\Big(\frac{A_n}{\sqrt{n}}\Big)\Big) - \mathbb{E}\Big[\mathrm{Tr}\Big(P_k\Big(\frac{A_n}{\sqrt{n}}\Big)\Big)\Big] = U_k^{(n)} - \mathbb{E}\left[U_k^{(n)}\right].$$

More precisely, their difference goes to zero in law. Since $\mathbb{E}\left[U_k^{(n)}\right]$ goes to zero by Proposition 4.2.14, we obtain that

$$\lim_{n\to\infty}(U_k^{(n)} - n^{-k/2}\hat{\mathcal{A}}_{c_k})_{k\geqslant 1} = 0$$

which completes the proof of Theorem 4.2.4.

□

4.3 Proof of Theorem 4.1.3

The goal of this section is to prove Theorem 4.1.3. Note that from Proposition 4.3.1, one can derive the result of Theorem 4.1.3 by using uniform asymptotics of Hermite polynomials. In particular, this approach does not require the use of Proposition 4.2.10. Since this alternative method involves more intricate computations, we opted for the approach presented in this section.

For $I \subset [n]$, we denote by A_I the submatrix of $A_{n,t}$ obtained by taking rows and columns with index in I. Since one can write

$$\det\left((1+tz^2) - z\frac{A_{n,t}}{\sqrt{n}}\right) = \sum_{k=0}^{n}(1+tz^2)^{n-k}(-z/\sqrt{n})^k S_k^{(n)} \tag{4.3.1}$$

where $S_k^{(n)} = \sum_{I\subset[n]:|I|=k}\det(A_I)$ is coefficient of w^k in the polynomial $\det(1+wA)$. The expectation $E[f_n(z)]$ depends only on t and be written in terms of Hermite polynomials, see Proposition 4.3.1 below.

Recall that if $(X_n)_{n\geqslant 1}$ is a uniformly integrable sequence of random variables which converge in law to X, then $\lim_{n\to\infty}\mathbb{E}[X_n] = \mathbb{E}[X]$. Since the second order moment of $\{f_n\}_{n\geqslant 1}$ is uniformly bounded on compact subsets by Proposition 4.2.10, the family is uniformly integrable and therefore for $z \in \mathbb{D}$,

$$\lim_{n\to\infty}\mathbb{E}[f_n(z)] = \mathbb{E}\left[e^{-F_t(z)}\right] \tag{4.3.2}$$

with

$$F_t(z) = \sum_{k\geqslant 1} X_k \frac{z^k}{\sqrt{k}}$$

for a family $(X_k)_{k\geqslant 1}$ of independent Gaussian random variables satisfying $\mathbb{E}[X_k] = 0$, $\mathbb{E}[X_k^2] = t^k$ and $\mathbb{E}[|X_k|^2] = 1$. Since $|\mathbb{E}[f_n(z)]| \leqslant \sqrt{\mathbb{E}[|f_n(z)|^2]}$ and since $\mathbb{E}[|f_n(z)|^2]$ is uniformly bounded on compact sets by Proposition 4.2.10, $\mathbb{E}[f_n(z)]$ is a precompact sequence of holomorphic functions by Montel's theorem. This implies that the convergence in (4.3.2) is uniform on compact sets. It is enough to compute

$$\mathbb{E}\left[e^{-F_t(z)}\right] = \mathbb{E}\left[e^{-\sum_{k\geqslant 1} X_k \frac{z^k}{\sqrt{k}}}\right] = e^{\frac{1}{2}\sum_{k\geqslant 1}\mathbb{E}[X_k^2]\frac{z^{2k}}{k}} = e^{\frac{1}{2}\sum_{k\geqslant 1} t^k \frac{z^{2k}}{k}} = \sqrt{1-tz^2}$$

which completes the proof of Theorem 4.1.3.

For the sake of completeness, we provide an explicit expression of the expectation $\mathbb{E}[f_n(z)]$ in Proposition 4.3.1.

Proposition 4.3.1 (Mean characteristic polynomial)**.** *Let $A_{n,t}$ be a random matrix as in Theorem 4.1.3. Then, for every $z \in \mathbb{D}$,*

$$\mathbb{E}[f_{n,t}(z)] = \mathrm{e}^{-ntz^2/2}\left(\sqrt{\frac{t}{n}}z\right)^n H_n\left(\sqrt{\frac{n}{t}}\left(\frac{1}{z}+tz\right)\right). \tag{4.3.3}$$

Remark 4.3.2 (Universality of the expectation)**.** Notice that the expectations involved in $\mathbb{E}[f_n(z)]$ only depend on $\mathbb{E}[a_{1,2}a_{2,1}] = t$. Therefore, $\mathbb{E}[f_{n,t}(z)]$ can be obtained by considering EGE matrices for the same t and is related to the kernel when we see the eigenvalues as a determinantal process. Namely, if $(Z_1,\dots,Z_n) \sim \frac{1}{\mathcal{Z}}\prod_{i<j}^n |z_j - z_i|^2 \mathrm{d}\mu^{\otimes n}(z_1,\dots,z_n)$, then

$$\mathbb{E}\left[\prod_{i=1}^n (z - Z_i)\right] = p_n(z),$$

where p_n is the monic orthogonal polynomial of degree n with respect to μ.

For convenience of the reader, we prefer to give a more direct proof of Proposition 4.3.1. Recall that $S_k^{(n)} = \sum_{I\subset[n]:|I|=k}\det(A_I)$ are the coefficients in (4.3.1).

Lemma 4.3.3 (Mean coefficient)**.** *For $1 \leqslant k \leqslant n$,*

$$\mathbb{E}\left[S_k^{(n)}\right] = \begin{cases} 0 \text{ if } k \in 2\mathbb{N}+1 \\ \binom{n}{k}(k-1)!!(-t)^{k/2} \text{ if } k \in 2\mathbb{N}. \end{cases} \tag{4.3.4}$$

where $(l)!! = l\cdot(l-2)\dots 3\cdot 1$ for $l \in 2\mathbb{N}+1$.

Proof. Let $I \subset [n]$ such that $|I| = k$. Then,

$$\mathbb{E}[\det(A_I)] = \sum_\sigma (-1)^\sigma \mathbb{E}\prod_{i\in I} a_{i,\sigma(i)}.$$

where the sum is over permutations of I. The expectation is non-zero if and only if σ is a product of transpositions as each term is centered and $(a_{i,j}, a_{j,i})$ is independent of $\{(a_{l,k}, a_{k,l}), (k,l) \neq (i,j)\}$. Thus, I has to be of even cardinal so that $k = 2l$ for some $l \geqslant 1$. There are $(2l-1)!! = (2l-1)(2l-3)\dots 1$ permutations that are product of transpositions since they are in bijection with pairings of $2l$ elements. Each such permutation gives a contribution of $(\mathbb{E}[a_{1,2}a_{2,1}])^l = t^l$ and has a signature of $(-1)^l$. Therefore,

$$\mathbb{E}[\det(A_I)] = (2l-1)!!(-t)^l,$$

which only depends on the cardinal of I. Thus,

$$\mathbb{E}\left[S_{2l}^{(n)}\right] = \binom{n}{2l}(2l-1)!!(-t)^l$$

$\square$

Proof of Proposition 4.3.1. Let $z \in \mathbb{D}$. Using (4.3.1), Lemma 4.3.3 and $\frac{(2k-1)!!}{(2k)!} = \frac{1}{2^k k!}$,

$$\begin{aligned}
\mathbb{E}[f_{n,t}(z)] &= \mathrm{e}^{-ntz^2/2} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (1+tz^2)^{n-2k} \frac{z^{2k}}{n^k} (2k-1)!!(-t)^k \\
&= \mathrm{e}^{-ntz^2/2} \left(\sqrt{\frac{t}{n}} z\right)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{2^k (n-2k)! k!} \left(\sqrt{\frac{n}{t}} \left(\frac{1}{z} + tz\right)\right)^{n-2k} \\
&= \mathrm{e}^{-ntz^2/2} \left(\sqrt{\frac{t}{n}} z\right)^n H_n \left(\sqrt{\frac{n}{t}} \left(\frac{1}{z} + tz\right)\right).
\end{aligned}$$

□

Chapter 5

Characteristic polynomial of Ewens random permutations

This chapter presents our results on the convergence of the characteristic polynomial in the context of generalized Ewens distributed permutations, which encompasses the uniform case of Theorem 2.2.7 of Coste, Lambert and Zhu [CLZ24] presented in Section 2.2.2 together with Ewens random permutations (1.1.12). The generalized Ewens distribution was introduced by Nikeghbali and Zeindler [NZ13] as a generalization of the classical Ewens distribution (1.1.12) by assigning different weights to each cycle lengths. Following the results of Chhaibi, Najnudel and Nikeghbali [CNN17] on the characteristic polynomial of Haar unitary matrices, Bahier [Bah19a] showed the convergence of the characteristic polynomial of Ewens permutation matrices at a microscopic scale around one and near irrational angles on the unit circle. Here, we consider the characteristic polynomial in a different regime namely inside the open unit disk where there are no eigenvalues.

5.1 The generalized Ewens measure

For $n \geqslant 1$, we denote by S_n the group of permutations of $\{1, \ldots, n\}$.

Definition 5.1.1 (Generalized Ewens measure, [NZ13]). Let $\Theta = (\theta_k)_{k\geqslant 1}$ be a sequence of positive real numbers. For $n \geqslant 1$, the *generalized Ewens measure* is the probability measure $\mathrm{d}\mathbb{P}_n^\Theta$ on S_n defined by

$$\mathrm{d}\mathbb{P}_n^\Theta[\sigma] = \frac{1}{n! h_n^\Theta} \prod_{k=1}^{n} \theta_k^{C_k(\sigma)} \tag{5.1.1}$$

where for a permutation $\sigma \in S_n$ and $k \geqslant 1$, $C_k(\sigma)$ is the number of cycles of σ with length k.

The Ewens measure corresponds to the case where the sequence Θ is constant equal to $\theta > 0$ in which case $h_n^\Theta = \binom{\theta+n-1}{n}$. The uniform measure on S_n corresponds to the Ewens distribution with parameter $\theta = 1$. From the sequence $\Theta = (\theta_k)_{k\geqslant 1}$, one defines as in [NZ13],

$$g_\Theta(z) = \sum_{k\geqslant 1} \frac{\theta_k}{k} z^k \text{ and } G_\Theta(z) = \exp(g_\Theta(z)) \tag{5.1.2}$$

as formal power series. For the Ewens measure of parameter θ, g_Θ and G_Θ are holomorphic in $\mathbb{D}$ with $g_\Theta(z) = -\theta \log(1-z)$ and $G_\Theta(z) = (1-z)^{-\theta}$. By [Hug+13, Lemma 2.6], one has

$$G_\Theta(z) = \sum_{n\geqslant 0} h_n^\Theta z^n,$$

where h_n^Θ are the constants in the definition of the generalized Ewens distribution (5.1.1).

In this chapter, we consider characteristic polynomials of random matrices associated to random permutations sampled from the generalized Ewens distribution. Since permutations $\sigma \in S_n$ can be viewed as permutation matrices of size n, we say that A_n follows the generalized Ewens distribution if it is the matrix obtained from a permutation σ sampled from (5.1.1). The characteristic polynomial $p_n(z) = \det(1 - zA)$ of a permutation matrix A can be expressed as

$$p_n(z) = \prod_{k=1}^{n} (1-z^k)^{C_k^{(n)}}, \tag{5.1.3}$$

where $C_k^{(n)}, 1 \leqslant k \leqslant n$ are the cycle lengths of the associated random permutation. Note that the eigenvalues of A_n are explicit and are given by roots of unity located on the unit circle. Our result aims at showing the convergence of $(p_n)_{n\geqslant 1}$ as a sequence of random holomorphic functions defined on the unit disk. As in [BCG22; FG23; Cos23] and [CLZ24], we consider the limit of the characteristic polynomial in the region outside of the eigenvalue support, namely $p_n(z) = z^n \det(z^{-1} - A_n)$ so that for $z \in \mathbb{D}$, z^{-1} lies outside of the unit circle and $\det(z^{-1} - A_n)$ does not vanish.

5.2 Convergence of the characteristic polynomial

5.2.1 Main result

For $n \geqslant 1$ and $\Theta = (\theta_k)_{k\geqslant 1}$ as above, we consider A_n the random matrix associated to a permutation σ sampled from (5.1.1). In this chapter, we consider characteristic polynomial

$$p_n(z) = \det(1 - zA_n) \tag{5.2.1}$$

inside the unit disk $z \in \mathbb{D} = \{x \in \mathbb{C} : |x| < 1\}$. Let us denote by $\mathcal{H}(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$ endowed with the topology of convergence on compact subsets of $\mathbb{D}$. Our main result is the convergence of p_n as a random variable in $\mathcal{H}(\mathbb{D})$ in law towards a limit function $F \in \mathcal{H}(\mathbb{D})$. The above convergence holds for parameters Θ such that the generating series g_Θ satisfies some conditions that we now define as Definition 5.2.1 which is an adaptation of a definition given in Section 5.2.1 of [Hwa94]. One can also find it as Definition 2.9 in [Hug+13] or Definition 2.8 in [NZ13].

Definition 5.2.1 (Logarithmic class function)**.** A function g is said to be in $F(r, \gamma, K)$ for $r > 0$, $\gamma \geqslant 0$ and $K \in \mathbb{C}$ if

- There exists $R > r$ and $\phi \in (0, \pi/2)$ such that g is holomorphic in $\Delta(r, R, \phi) \setminus \{r\}$ where $\Delta(r, R, \phi) = \{z \in \mathbb{C} : |z| \leqslant R, |\arg(z - r)| \geqslant \phi\}$.
- As $z \to r$, $g(z) = -\gamma \log(1 - z/r) + K + O(z - r)$.

In the case of the Ewens measure of parameter θ, we have $g_\Theta(z) = -\theta \log(1-z)$ so that $g_\Theta \in F(1, \theta, 0)$. Note that if $\gamma > 0$, the parameter r is unique.

Our main result is Theorem 5.2.2 which gives the convergence of the characteristic polynomial towards a limit function for sequences Θ such that g satisfies the conditions of Definition 5.2.1.

Theorem 5.2.2 (Convergence of the characteristic polynomial)**.** *Let $\Theta = (\theta_k)_{k\geqslant 1}$ be a sequence of positive real numbers such that $g_\Theta \in F(r,\gamma,K)$ for $r>0$ and $\gamma>0$. We have the convergence in law, for the topology of local uniform convergence in $\mathbb{D}$*

$$p_n \xrightarrow[n\to\infty]{\text{law}} F \tag{5.2.2}$$

where

$$F(z) = \exp\left(-\sum_{k\geqslant 1}\frac{z^k}{k}X_k\right), \quad X_k = \sum_{\ell | k}\ell Y_\ell, \tag{5.2.3}$$

with $(Y_\ell)_{\ell\geqslant 1}$ independent Poisson random variables with parameter $\frac{\theta_\ell}{\ell}r^\ell$.

The previous theorem gives in particular the convergence of the characteristic polynomial for Ewens permutation matrices. Indeed, for constant θ, the function $g_\Theta \in F(1,\theta,0)$ so that p_n converges towards the limit function as conjectured in [CLZ24].

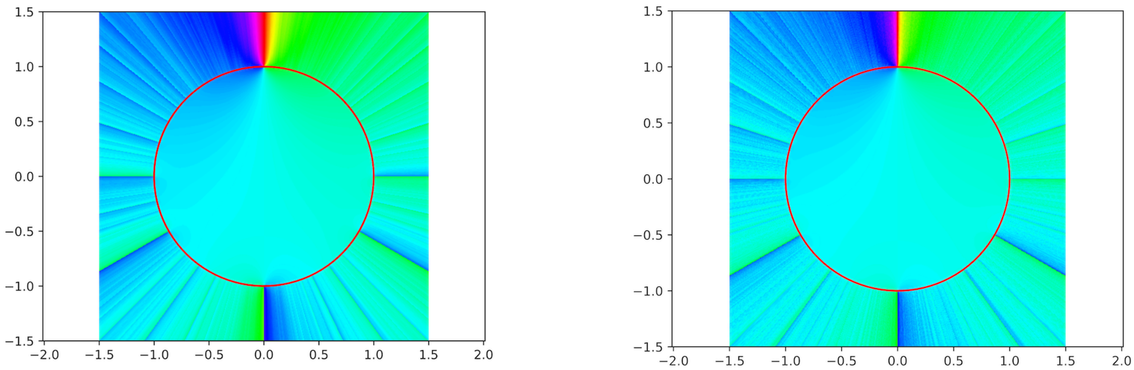

Figure 5.1: Phase portrait of p_n for an Ewens matrix of size $n = 10000$ with parameter $\theta = 100$ (left) and phase portrait of the limit function with same parameter (right). The unit circle is represented in red.

Remark 5.2.3 (Outside region)**.** Theorem 5.2.2 deals with the convergence in law for $z \in \mathbb{D}$ so that $p_n(z) = \det(1 - zA_n)$ does not vanish as eigenvalues of A_n are located on the unit circle. One can extend the previous to values of $p_n(z)$ for z in $\mathbb{C}\backslash\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| > 1\}$ under suitable normalization. Indeed, notice that the generalized Ewens distribution (5.1.1) is invariant under inversion, that is, if σ has distribution (5.1.1) then so does σ^{-1} as they both have the same cycle structure. Thus, $A_n = A_n^{-1}$ in law. Furthermore, for $z \in \mathbb{C}\setminus\overline{\mathbb{D}}$, $\det(1 - zA_n) = (-z)^n \det(A_n)\det(1 - z^{-1}A_n^{-1})$ so that if $\tilde{p}_n(z) = \frac{p_n(z)}{(-z)^n\det(A_n)}$ and $\iota(z) = \frac{1}{z}$, Theorem 5.2.2, gives the convergence in law on $\mathbb{C}\setminus\overline{\mathbb{D}}$ of $\tilde{p}_n$ to $F\circ\iota$.

5.2.2 Method of proof

The proof of Theorem 5.2.2 relies on the same structure as in [BCG22] and [FG23], which is recalled in Lemma 5.3.1. This lemma corresponds to Lemma 4.2.1 from the study of Gaussian elliptic matrices and we restate it below for clarity. It is a consequence of the general fact stated in [Shi12] that a tight sequence of holomorphic functions in $\mathcal{H}(\mathbb{D})$ whose coefficients convergence in law for finite dimensional distributions converges to a random analytic function. We first show that the sequence $(p_n)_{n\geqslant 1}$ is tight which is Theorem 5.3.2 proved in section 5.4. The question of tightness for the Ewens model was raised in [CLZ24]. Here, tightness is archieved by a uniform control of the second moment of p_n. This control relies on results from Hwang [Hwa94] on singularity analysis for generating functions. The finite dimensional convergence of coefficients is obtained by showing the convergence of traces of powers, see the discussion above Theorem 5.3.3. The convergence of traces for generalized Ewens matrices was done in [Hug+13] and [NZ13]. We recall their results in Section 5.5 where Theorem 5.3.3 is proved. From these two results, one is able to derive the convergence of p_n towards a random analytic function F. The fact that F coincides with the exponential of a Poisson series is the purpose of Theorem 5.3.4 proved in section 5.6. In the rest of the chapter, we assume that Θ is fixed and we write g and G for the functions defined in (5.1.2) for notation convenience.

5.3 Proof of the convergence

Recall that $\mathcal{H}(\mathbb{D})$ denotes the space of analytic functions on $\mathbb{D}$ endowed with the topology of local uniform convergence. In order to show the convergence in law of a sequence $(f_n)_{n\geqslant 1}$ in $\mathcal{H}(\mathbb{D})$, we rely on Lemma 5.3.1 which is close to Proposition 2.5 in [Shi12]. It is also stated as Lemma 3.2 in [BCG22] and proved therein.

Lemma 5.3.1 (Tightness and convergence of coefficients imply convergence of functions)**.** *Let $\{f_n\}_{n\geqslant 1}$ be a sequence of random elements in $\mathcal{H}(\mathbb{D})$ and denote the coefficients of f_n by $(\xi_k^{(n)})_{k\geqslant 0}$ so that for all $z \in \mathbb{D}$, $f_n(z) = \sum_{k\geqslant 0} \xi_k^{(n)} z^k$. Suppose also that the following conditions hold.*

(a) The sequence $\{f_n\}_{n\geqslant 1}$ is a tight sequence of random elements of $\mathcal{H}(\mathbb{D})$.

(b) There exists a sequence $(\xi_k)_{k\geqslant 0}$ of random variables such that, for every $m \geqslant 0$, the vector $(\xi_0^{(n)}, \dots, \xi_m^{(n)})$ converges in law as $n \to \infty$ to $(\xi_0, \dots, \xi_m)$.

Then, $f(z) = \sum_{k\geqslant 0} \xi_k z^k$ is a well-defined function in $\mathcal{H}(\mathbb{D})$ and f_n converges in law towards f in $\mathcal{H}(\mathbb{D})$ for the topology of local uniform convergence.

In Lemma 5.3.1, there are two topologies involved, namely the topology of uniform convergence on compact subsets of the unit disk for the space $\mathcal{H}(\mathbb{D})$ and the weak topology, or the convergence in law, in the probability space. We thus need to show that the sequence $(p_n)_{n\geqslant 1}$ is tight and then study the limit of finite dimensional distributions for its coefficients. The first part is given by Theorem 5.3.2 which is proved in section 5.4.

Theorem 5.3.2 (Tightness)**.** *The sequence $(p_n)_{n\geqslant 1}$ is tight in $\mathcal{H}(\mathbb{D})$.*

It remains to study the coefficients of p_n. Let us write

$$p_n(z) = 1 + \sum_{k=1}^{n} (-z)^k \Delta_k(A_n)$$

where $\Delta_k(A)$ is the coefficient of z^k in $\det(1 + zA)$. Coefficients $\Delta_k(A_n)$ can be expressed via traces $\left(\mathrm{Tr}\left[A_n^\ell\right]\right)_{1\leqslant \ell\leqslant k}$ so that

$$\Delta_k(A_n) = \frac{1}{k!} P_k\left(\mathrm{Tr}\left[A_n^1\right], \dots, \mathrm{Tr}\left[A_n^k\right]\right)$$

where the polynomials P_k do not depend on n. In order to study the convergence in law of coefficients $(\Delta_1(A_n), \dots, \Delta_k(A_n))$, it suffices to study the convergence of traces $\left(\mathrm{Tr}[A_n^1], \dots, \mathrm{Tr}[A_n^k]\right)$ which is given by Theorem 5.3.3. Recall that r denotes the radius of convergence of g, see Definition 5.2.1.

Theorem 5.3.3 (Convergence of coefficients). *Let $k \geqslant 1$ and assume that $g_\Theta \in F(r, \gamma, K)$ for $r > 0$ and $\gamma > 0$. We have the convergence in law as $n \to \infty$,*

$$\left(\mathrm{Tr}[A_n], \dots, \mathrm{Tr}\left[A_n^k\right]\right) \to (X_1, \dots, X_k) \tag{5.3.1}$$

where

$$X_k = \sum_{\ell | k} \ell Y_\ell \tag{5.3.2}$$

with $(Y_\ell, \ell \geqslant 0)$ are independent Poisson random variables with parameter $\frac{\theta_d}{d} r^d$.

Thanks to Lemma 5.3.1, Theorem 5.3.2 and Theorem 5.3.3, we derive that p_n converges towards the random analytic function $F \in \mathcal{H}(\mathbb{D})$ given by

$$F(z) = 1 + \sum_{k \geqslant 1} \frac{(-z)^k}{k!} P_k(X_1, \dots, X_k).$$

To obtain the expression of Theorem 5.2.2, we rely on Theorem 5.3.4, proved in Section 5.6 which yields the desired expression and ends the proof of Theorem 5.2.2.

Theorem 5.3.4 (Poisson expression for F). *For every $z \in \mathbb{D}$, one has almost surely,*

$$F(z) = \exp(-f(z)) \tag{5.3.3}$$

where

$$f(z) = \sum_{k \geqslant 1} \frac{X_k}{k} z^k$$

and where $(X_k)_{k \geqslant 1}$ are defined as in Theorem 5.2.2.

5.4 Tightness

The goal of this section is to prove Theorem 5.3.2. We start by Lemma 5.4.1 which reduces the tightness of a sequence of a functions (f_n) to proving tightness of their local supremum. This Lemma corresponds to Proposition 2.5 of [Shi12].

Lemma 5.4.1 (Reduction to uniform control). *Let $(f_n)_{n \geqslant 1}$ be a sequence of random elements of $\mathcal{H}(\mathbb{D})$. If for every compact $K \subset \mathbb{D}$, the sequence $(\sup_{z \in K} |f_n(z)|)_{n \geqslant 1}$ is tight, then $(f_n)_{n \geqslant 1}$ is tight.*

It therefore suffices to show that $(\sup_{z \in K} |f_n(z)|)_{n \geqslant 1}$ is tight. By subharmonicity of $|f_n(z)|^2$, this is equivalent to show that $(\sup_{z \in K} \mathbb{E}[|f_n(z)|^2])_{n \geqslant 1}$ is bounded, see for instance [Shi12, Lemma 2.6]. We will show this for the sequence $(p_n)_{n \geqslant 1}$ of characteristic polynomials by giving a uniform control of the second moment of p_n which is Proposition 5.4.2. This control comes from an asymptotic given in Corollary 3.8 of [Hug+13] where we explicit the fact that the error is uniform for z in compact subsets of $\mathbb{D}$.

Recall that the functions g_Θ and G_Θ are defined in (5.1.2) by $g_\Theta = \sum_{k \geqslant 1} \frac{\theta_k}{k} z^k$ and $G_\Theta(z) =$

$\exp(g_\Theta(z))$ for $|z| < r$ where r is the radius of convergence of g_Θ. For notation convenience, we will denote them by g and G without specifying the parameter Θ. For $\delta > 0$, let us denote by

$$\mathbb{D}_\delta = \{z \in \mathbb{C} \mid |z| < \delta\}$$

the open disk of radius δ.

Proposition 5.4.2 (Second moment control)**.** *Assume that $g \in F(r, \gamma, K)$ for some $r > 0$, $\gamma \geqslant 0$ and $K \in \mathbb{C}$. For $\delta \in (0,1)$ and $z \in \mathbb{D}_\delta$, one has the asymptotic expansion*

$$\mathbb{E}[|p_n(z)|^2] = \frac{G(r|z|^2)}{G(rz)G(r\overline{z})} + O\left(\frac{1}{n}\right) \tag{5.4.1}$$

where the O term holds uniformly in $z \in \mathbb{D}_\delta$.

Proof. Let $\delta \in (0,1)$. For $z \in \mathbb{D}_\delta$, one has using Corollary 3.6 of [Hug+13]

$$\sum_{n \geqslant 0} t^n h_n \mathbb{E}[|p_n(z)|^2] = \exp(g(t)) S_z(t) \tag{5.4.2}$$

where h_n are the coefficients of (5.1.1) and

$$S_z(t) = \frac{G(t|z|^2)}{G(tz)G(t\overline{z})}. \tag{5.4.3}$$

We now apply the method of [Hwa94] to $\exp(g(t))S_z(t)$ as done in Section 5.3.2 therein.

For every $z \in \mathbb{D}_\delta$, the function $t \mapsto G(zt)$ is analytic for $|t| \leqslant r + \epsilon_1$ for some $\epsilon_1 > 0$ such that $\delta(r + \epsilon_1) < r$ since $G(u) = \exp(g(u))$ and since that g is analytic in $\mathbb{D}_r$. Therefore, for every $z \in \mathbb{D}_\delta$, $t \mapsto S_z(t)$ is analytic for $|t| \leqslant r + \epsilon_1$.
By assumption, g is analytic in $\Delta(r, r + \epsilon_2, \phi) \setminus \{r\}$ for some $\epsilon_2 > 0$ and $0 < \phi < \frac{\pi}{2}$. Set $R = r + \min(\epsilon_1, \epsilon_2)$ and $\xi > 0$ such that $r\mathrm{e}^\xi < R$. As in the proof of Theorem 12 in [Hwa94], we write

$$h_n \mathbb{E}[|p_n(z)|^2] = \frac{1}{2i\pi} \int_\Gamma \frac{\exp(g(t))S_z(t)}{t^{n+1}} dt + \frac{1}{2i\pi} \int_{\Gamma'} \frac{\exp(g(t))S_z(t)}{t^{n+1}} dt$$

where

$$\Gamma = \{t \in \mathbb{C} \mid |t - 1| = r(\mathrm{e}^\xi - 1), |\arg(t - r)| \geqslant \phi\}$$
$$\Gamma' = \{t \in \mathbb{C} \mid |t| = r\mathrm{e}^\xi, |\arg(t - r)| \geqslant \phi\}.$$

For the second integral over Γ', we may use that

$$|S_z(t)| \leqslant \frac{\sup_{|u| \leqslant \delta^2 r\mathrm{e}^\xi} |G(u)|}{(\inf_{|u| \leqslant \delta r\mathrm{e}^\xi} |G(u)|)^2} = C \tag{5.4.4}$$

where C does not depend on z. The contribution of this integral is $O(r^{-n}\mathrm{e}^{-n\xi})$ as in [Hwa94] and where the O term is uniform in z. The asymptotic of the integral over Γ involves the function S_z only via $U_z(t)$ where

$$U_z(t) = h(t) S_z(r\mathrm{e}^{-t})$$

with h defined with the parameters relative to g only. The asymptotic in [Hwa94] relies on the asymptotic development $U_z(t) = S_z(r) + O(|t|)$. For our concerns, we check that the error term is uniform in z. We have

$$U_z(t) = h(t)(S_z(r) + O(|t|))$$

where the O is uniform in z since the constant can be taken as $\sup_{|t|\leqslant r} |S'_z(t)|$ which can be bounded uniformly with respect to z by bounding values of G and G' in $\mathbb{D}_{r\delta}$ in a similar fashion as in (5.4.4). Since $h(t) = (1 + O(|t|))$ does not depend on z, we derive that $U_z(t) = S_z(r) + O(|t|)$ uniformly in $z \in \mathbb{D}_\delta$. The rest of the proof of [Hwa94] applies so that one derives the same asymptotic (5.4.1) with an error tern uniform in $z \in \mathbb{D}_\delta$. $\square$

From Proposition 5.4.2, one derives that $\mathbb{E}[|p_n(z)|^2]$ is bounded by a deterministic function of z that does not depend on n so that the sequence $(p_n)_{n\geqslant 1}$ is tight which ends the proof of Theorem 5.3.2.

5.5 Convergence of traces

The purpose of this section is to prove Theorem 5.3.3 on the finite dimensional convergence for traces of monomials $A_n^1, \dots, A_n^k$. The study of the convergence of traces for random permutation matrices following the generalized Ewens distribution has been done in [NZ13]. The convergence of finite dimensional distributions is a consequence of a functional equality on generating functions, see (5.5.1) below which is Theorem 3.1 of [NZ13],

$$\sum_{n\geqslant 0} h_n \mathbb{E}\left[\exp\left(i\sum_{m=1}^{b} s_m C_m^{(n)}\right)\right] t^n = \exp\left(\sum_{m=1}^{b} \frac{\theta_m}{m}(\mathrm{e}^{is_m} - 1)t^m\right) G(t). \tag{5.5.1}$$

From (5.5.1), using the result of [Hwa94] on singularity analysis for generating functions, one derives as done in [NZ13, Corollary 3.2], that for every $k \geqslant 1$, the convergence in law

$$\left(C_1^{(n)}, \dots, C_k^{(n)}\right) \to (Y_1, \dots, Y_k)$$

holds with $(Y_\ell)_{\ell\geqslant 1}$ independent Poisson random variables with parameter $\frac{\theta_\ell}{\ell} r^\ell$. Using that

$$\mathrm{Tr}\left[A_n^k\right] = \sum_{\ell | k} \ell C_\ell^{(n)}$$

yields the result of Theorem 5.3.3 by the Cramer-Wold theorem.

5.6 Poisson Expression for the limit

The goal of this section is to derive the expression of the limiting function as the exponential of a Poisson series as stated in Theorem 5.2.2. For the sake of completeness, we provide another representation for the limit function of Theorem 5.2.2. This representation given in Lemma 5.6.2 has the form of an infinite product and is inspired from [Cos23] where the *Poisson multiplicative function*, that is, the exponential of a Poisson series appeared in the context of Bernoulli matrices as presented in Section 2.3.3.

Let $(Y_\ell)_{\ell\geqslant 1}$ be a family of independent Poisson random variables with parameters $(\frac{r^\ell\theta_\ell}{\ell})_{\ell\geqslant 1}$. Consider the power series

$$f(z)=\sum_{k\geqslant 1}\frac{X_k}{k}z^k,\ \text{ where } X_k=\sum_{\ell|k}\ell Y_\ell.$$

Recall that r is the radius of convergence of the series $g(z)=\sum_{k\geqslant 1}\frac{\theta_k}{k}z^k$ so that $\frac{1}{r}=\limsup_k\theta_k^{\frac{1}{k}}$. We first show that f is a well-defined function on the open disk $\mathbb{D}$ in Proposition 5.6.1. Computation of convergence radius for Poisson series were done in [CLZ24] for independent Poisson variables $(Y'_\ell)_{\ell\geqslant 1}$ with parameters $(\frac{d^\ell}{\ell})_{\ell\geqslant 1}$. In particular, for $d>1$, the radius of convergence of $f'=\sum_{k\geqslant 1}\frac{X'_k}{k}z^k$ with $X'_k=\sum_{\ell|k}\ell Y'_\ell$ is almost surely equal to $\frac{1}{d}$, see Theorem 2.7 in [CLZ24].

Proposition 5.6.1 (Radius of convergence for limit function)**.** *Almost surely, the radius of convergence of f is greater than 1.*

Proof. To find the radius of convergence of f, one must compute $\limsup(\frac{X_k}{k})^{\frac{1}{k}}=\limsup X_k^{\frac{1}{k}}$. Let $\epsilon>0$. There exists ℓ_0 such that for $\ell\geqslant\ell_0$,

$$\left|\frac{1}{r}-\sup_{\ell\geqslant\ell_0}\theta_\ell^{\frac{1}{\ell}}\right|\leqslant\frac{\epsilon}{r}$$

so that for $\ell\geqslant\ell_0$,

$$r^\ell\theta_\ell\leqslant(1+\epsilon)^\ell.$$

Define on the same probability space sequences $(Y_\ell)_{\ell\geqslant 1}$ and $(Y'_\ell)_{\ell\geqslant 1}$ having respective parameters $(\frac{r^\ell\theta_\ell}{\ell})_{\ell\geqslant 1}$ and $(\frac{d^\ell}{\ell})_{\ell\geqslant 1}$, such that $Y_\ell\leqslant Y'_\ell$ almost surely for $\ell\geqslant\ell_0$. Then, almost surely,

$$X_k\leqslant X'_k+\sum_{\substack{\ell|k\\ \ell\leqslant\ell_0}}\ell(Y_\ell-Y'_\ell)$$

where $X'_k=\sum_{\ell|k}\ell Y'_\ell$. We have that $\sum_{\substack{\ell|k\\ \ell\leqslant\ell_0}}\ell(Y_\ell-Y'_\ell)\leqslant\sum_{\ell=1}^{\ell_0}\ell(Y_\ell-Y'_\ell)=c$ where c is a random constant that does not depend on k so that almost surely,

$$\limsup X_k^{\frac{1}{k}}\leqslant\limsup\left(X'_k+c\right)^{\frac{1}{k}}\leqslant\limsup(X'_k)^{\frac{1}{k}}=1+\epsilon$$

where we have used that $X'_k+c\leqslant X'_k(1+|c|)$ for the second inequality and that the convergence radius of $\sum_{k\geqslant 1}\frac{X'_k}{k}z^k$ is almost surely $\frac{1}{1+\epsilon}$ using Theorem 2.7 of [CLZ24]. Therefore, we have that, for every $\epsilon>0$, the convergence radius r_f of f satisfies

$$r_f\geqslant\frac{1}{1+\epsilon},$$

so that $r_f\geqslant 1$ almost surely. □

Since $F(0)=1$ and that $F\in\mathcal{H}(\mathbb{D})$, one can consider $\log(F)$ which is a well-defined analytic function in a neighborhood of the origin, where log is the principal branch of the logarithm. This function coincides with $-f$ so that they are both equal. Both functions are well-defined in the unit disk from which one derives the desired expression of Theorem 5.3.4.

Lemma 5.6.2 (Infinite product expression)**.** *For $z \in \mathbb{D}$, one has*

$$\exp(-f(z)) = \prod_{k\geqslant 1} \left(1 - z^k\right)^{Y_k}. \tag{5.6.1}$$

Proof. The expression above is due to the inversion

$$\sum_{k\geqslant 1} \frac{X_k}{k} z^k = \sum_{\ell\geqslant 1} \ell Y_\ell \sum_{k\geqslant 1} \frac{z^{k\ell}}{k\ell} = -\sum_{\ell\geqslant 1} Y_\ell \log(1 - z^\ell),$$

which can be performed since uniform convergence holds for $z \in \mathbb{D}$. $\square$

This provides an example of log-correlated field as correlations for such function r are given by $\mathbb{E}[r(z)\overline{r(w)}] = -\log(1 - z\overline{w})$. For the generalized Ewens measure, the correlations are given by the generating function g as stated in Lemma 5.6.3.

Lemma 5.6.3 (Correlations of Poisson field)**.** *For $z, w \in \mathbb{D}$, one has*

$$Cov(f(z), f(w)) = \sum_{a,b\geqslant 1} \frac{1}{ab} g(rz^a \overline{w}^b). \tag{5.6.2}$$

Proof. Since we want to compute correlations, we must consider the series

$$\sum_{k\geqslant 1} \frac{X_k - \mathbb{E}[X_k]}{k} z^k. \tag{5.6.3}$$

From Proposition 5.6.1, we know that the convergence radius of $\sum_{k\geqslant 1} \frac{X_k}{k} z^k$ is at least 1. Let us check that the same holds for $\sum_{k\geqslant 1} \frac{\mathbb{E}[X_k]}{k} z^k$ so that (5.6.3) is well-defined for $z \in \mathbb{D}$. Let $\epsilon > 0$. As in Proposition 5.6.1, there exists $\ell_0 \geqslant 1$ such that for $\ell \geqslant \ell_0 : \theta_\ell r^\ell \leqslant (1+\epsilon)^\ell$. Thus,

$$\mathbb{E}[X_k] = \sum_{\ell | k} r^\ell \theta_\ell \leqslant \sum_{\substack{\ell | k \\ \ell\leqslant\ell_0}} (r^\ell \theta_\ell - (1+\epsilon)^\ell) + \sum_{\ell | k} (1+\epsilon)^\ell$$

so that $|\mathbb{E}[X_k]| \leqslant (c+1)\tau_k$ where $\tau_k = \sum_{\ell | k} (1+\epsilon)^k$ and $c = \sum_{\ell\leqslant\ell_0} |\theta_\ell r^\ell - (1+\epsilon)^\ell|$. The latter implies that $\limsup_k |\mathbb{E}[X_k]|^{1/k} \leqslant \limsup \tau_k^{1/k} = 1+\epsilon$ from which one derives that the convergence radius of $\sum_{k\geqslant 1} \frac{\mathbb{E}[X_k]}{k} z^k$ is greater than $\frac{1}{1+\epsilon}$. Since ϵ was arbitrary, the convergence radius is greater or equal to one so that (5.6.3) is well-defined for $z \in \mathbb{D}$. For

$z, w \in \mathbb{D}$, we thus compute

$$
\begin{aligned}
Cov(f(z), f(w)) &= \sum_{k,h} \frac{z^k \overline{w}^h}{kh} \sum_{\substack{i|h \\ j|k}} ij Cov(Y_i, Y_j) \\
&= \sum_{k,h} \frac{z^k \overline{w}^h}{kh} \sum_{\ell|k, \ell|h} \ell^2 Var[Y_\ell] \\
&= \sum_{k,h} \frac{z^k \overline{w}^h}{kh} \sum_{\ell|k, \ell|h} \ell \theta_\ell r^\ell \\
&= \sum_{\ell \geqslant 1} \frac{\theta_\ell r^\ell}{\ell} \sum_{a,b \geqslant 1} \frac{z^{a\ell} \overline{w}^{b\ell}}{ab} \\
&= \sum_{a,b \geqslant 1} \frac{1}{ab} \sum_{\ell \geqslant 1} \frac{\theta_\ell}{\ell} (rz\overline{w})^\ell \\
&= \sum_{a,b \geqslant 1} \frac{1}{ab} g(rz^a \overline{w}^b).
\end{aligned}
$$

□

Remark 5.6.4. In the case of uniform permutations [CLZ24] or even for Ewens random permutations (1.1.12), that is, $\theta_k = \theta$ for some $\theta > 0$, one has $r = 1$ and $g(z) = -\theta \log(1-z)$ so that

$$
Cov(f(z), f(w)) = -\theta \sum_{a,b \geqslant 1} \frac{1}{ab} \log\left(1 - z^a \overline{w}^b\right)
$$

which is the analog of the log-correlations obtained for the Gaussian holomorphic chaos. In general, the correlations for arbitrary sequences Θ are given by g. Moreover, the expectation of the limit can be expressed using G for any $z \in \mathbb{D}$,

$$
\mathbb{E}\left[\prod_{k \geqslant 1} \left(1 - z^k\right)^{Y_k}\right] = \prod_{k \geqslant 1} \mathrm{e}^{-\theta_k \frac{r^k z^k}{k}} = \frac{1}{G(rz)}.
$$

Part III

Articles on products of unitary matrices

Chapter 6

Positive formula for the product of conjugacy classes

In the framework of compact Lie groups, the problem of finding positive combinatorial formulas for the expansion of characters has been solved in the case of the unitary group $\mathrm{U}(n)$ by Littlewood and Richardson [LR34] and in full generality by Luzstig [Lus10], Kashiwara [KN94] and Littelmann [Lit94]. The case of conjugacy classes is still open in full generality. The goal of this chapter is to present our results which address this problem in the case of $\mathrm{U}(n)$. Conjugacy classes of $\mathrm{U}(n)$ are indexed by the symmetrized torus $\mathcal{H} = (\mathbb{R}^n/\mathbb{Z}^n)/S_n$, and the decomposition of the product of two conjugacy classes α and β is described by a probability distribution $\mathbb{P}[\cdot|\alpha,\beta]$ on $\mathcal{H}$. As for any compact Lie group, $\mathbb{P}[\cdot|\alpha,\beta]$ can be expressed in a weak sense as a complex weighted sum of characters, each of which being seen as a function on $\mathcal{H}$, see (6.2.3). When the conjugacy classes α and β have maximal dimension, in which case α and β are called regular, $\mathbb{P}[\cdot|\alpha,\beta]$ has a density $\mathrm{d}\mathbb{P}[\cdot|\alpha,\beta]$ with respect to the Lebesgue measure on $\mathcal{H}$. This chapter provides a positive formula for this density as a subtraction-free sum of volumes of some explicit polytopes.

The convolution of conjugacy classes of a compact Lie group G is also intimately related to the moduli spaces of G-valued flat connections on punctured Riemannian surfaces as presented in Section 3.2.5. The formula we obtain for the convolution of conjugacy classes directly translates into a simple and manifestly positive expression for the volume of the moduli space of flat connections on the three-punctured sphere for $G = \mathrm{SU}(n)$. This is up to our knowledge the first expression of those volumes as the volume of explicit polytopes.

This chapter is structured as follows. In Section 6.1, we present the main results of this chapter, namely Theorem 6.1.5 and Corollary 6.1.6. The first step of the proof is the semi-classical approximation of the density of the convolution product by a limit of quantum Littlewood-Richardson coefficients. Such approximation scheme is done in Section 6.2. All the work of the remaining part of this chapter consists in turning known expressions for the quantum Littlewood-Richardson coefficients into integers points counting in convex bodies, for which the convergence towards volumes of polytopes is straightforward, see [KT99]. Section 6.3 introduces the puzzle expression for those coefficients obtained in [Buc+16] and gives a first simplification of the puzzle formulation by only keeping the position of certain pieces of the puzzles. It is then deduced in Section 6.4 an expression of the coefficients as the counting of integers points in a family of convex polytopes indexed by certain two-colored tilings which are reminiscent of Figure 6.4, see Theorem 6.4.3 and

Corollary 6.4.4. Up to this point, those polytopes are degenerated and non-rational polytopes in a higher dimensional space, which prevents any proper asymptotic counting in the semi-classical limit (a similar problematic situation already occurred with Berenstein-Zelevinsky polytopes in the co-adjoint case, leading to the hive formulation of Knutson and Tao, see [KT99]). By a combinatorial work on the underlying two-colored tiling, we give in Section 6.5 a parametrization of the integer points of those polytopes in terms of integer points of genuine convex bodies. Remark that the results of Section 6.4 hold more generally for any coefficient of the two-step flag variety, a fact which is not true anymore from Section 6.5. The asymptotic counting of integers points in convex bodies is then much more tractable, and the conclusion of the proof of Theorem 6.1.5 and Corollary 6.1.6 is done in Section 6.6.

6.1 A positive formula for the density

Let $n \geqslant 3$. The set of conjugacy classes of $\mathrm{U}(n)$ is homeomorphic to the quotient space $\mathcal{H} = (\mathbb{R}^n/\mathbb{Z}^n)/S_n$, where the symmetric group S_n acts on $\mathbb{R}^n/\mathbb{Z}^n$ by permutation of the coordinates. This quotient space is described by the set of non-increasing sequences of $[0,1[^n$. For $\theta = (\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n) \in \mathcal{H}$, denote by $\mathcal{O}(\theta)$ the corresponding conjugacy class defined by

$$\mathcal{O}(\theta) := \left\{ U e^{2i\pi\theta} U^*,\, U \in \mathrm{U}(n) \right\}, \text{ where } e^{2i\pi\theta} = \begin{pmatrix} e^{2i\pi\theta_1} & 0 & \cdots & \\ 0 & e^{2i\pi\theta_2} & & \\ \vdots & & \ddots & \\ & & & e^{2i\pi\theta_n} \end{pmatrix}.$$

The product structure on $\mathrm{U}(n)$ translates into a convolution product $* : \mathcal{M}_1(\mathcal{H}) \times \mathcal{M}_1(\mathcal{H}) \to \mathcal{M}_1(\mathcal{H})$ on the space of probability distributions on $\mathcal{H}$ such that for $\theta, \theta' \in \mathcal{H}$, $\delta_\theta * \delta_{\theta'}$ is the distribution of $p(U_\theta U_{\theta'})$, where U_θ (resp. $U_{\theta'}$) is sampled uniformly on $\mathcal{O}(\theta)$ (resp. $\mathcal{O}(\theta')$) and $p : \mathrm{U}(n) \to \mathcal{H}$ maps an element of $\mathrm{U}(n)$ to its conjugacy class in $\mathcal{H}$.

Let us denote by $\mathcal{H}_{reg} = \{\theta \in \mathcal{H}, \theta_1 > \theta_2 > \ldots > \theta_n\}$ the set of regular conjugacy classes of $\mathrm{U}(n)$, namely the ones of maximal dimension in $\mathrm{U}(n)$. For $\alpha, \beta \in \mathcal{H}_{reg}$, $\delta_\alpha * \delta_\beta$ admits a density $\mathrm{d}\mathbb{P}[\cdot|\alpha,\beta]$ with respect to the Lebesgue measure on

$$\left\{ \gamma \in \mathcal{H} \mid \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i - \sum_{i=1}^n \gamma_i \in \mathbb{N} \right\}.$$

See Section 6.2 for a concrete proof of this classical result.

The toric hive cones $\mathcal{C}_g$

The main result of this chapter is a positive formula for $\mathrm{d}\mathbb{P}[\cdot|\alpha,\beta]$ in terms of the volume of polytopes similar to the hive model of Knutson and Tao [KT99]. For $0 \leqslant d \leqslant n$, define the *toric hive* $R_{d,n}$ as the set

$$R_{d,n} := \left\{ (v_1, v_2) \in [\![0,n]\!]^2, d \leqslant v_1 + v_2 \leqslant n + d \right\},$$

which can be represented as a discrete hexagon through the map $(v_1, v_2) \mapsto v_1 + v_2 e^{i\pi/3}$, see Figure 6.1 for a particular case and its hexagonal representation.

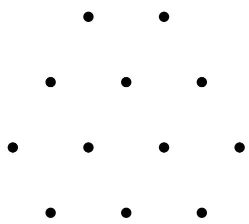

Figure 6.1: The set $R_{1,3}$ represented through the map $(v_1, v_2) \mapsto v_1 + v_2 e^{i\pi/3}$.

Boundary of the toric hive

For any set S and any function $f : R_{d,n} \to S$, we denote by f^A (resp f^B, f^C) the vector $(f((d-i) \vee 0, (n+d-i) \wedge n)_{0 \leqslant i \leqslant n}$ (resp. $(f(n+d-i \wedge n, i))_{0 \leqslant i \leqslant n}$, resp. $(f(n-i, i+d-n \vee 0))_{0 \leqslant i \leqslant n}$). The vectors f^A, f^B and f^C correspond respectively to the north-west, east and south-west boundaries of $R_{d,n}$ through the hexagonal representation, see Figure 6.2.

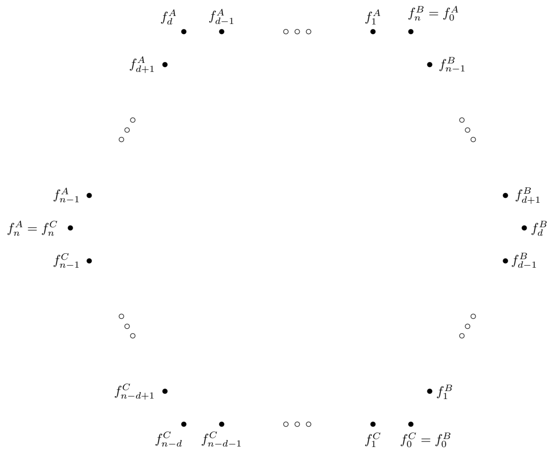

Figure 6.2: The set boundary vectors f^A, f^B and f^C.

Toric rhombus concavity

Let us call a *lozenge* of $R_{d,n}$ any sequence $(v^1, v^2, v^3, v^4) \in (R_{d,n})^4$ corresponding to one of the three configurations of Figure 6.3 in the hexagonal representation (in which $|v^i - v^{i+1}| = 1$ for $1 \leqslant i \leqslant 3$).

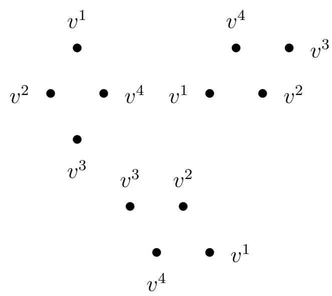

Figure 6.3: The three possible lozenges (v^1, v^2, v^3, v^4) (the position of the vertices can not be permuted).

Definition 6.1.1 (Regular labeling)**.** A function $g : R_{d,n} \to \mathbb{Z}_3$ is called a *regular labeling* whenever

- $g_i^A = n + i[3]$, $g_i^B = i[3]$ and $g_i^C = i[3]$,
- on any lozenge $\ell = (v^1, v^2, v^3, v^4)$,

$$\left(g(v^2) = g(v^4)\right) \Rightarrow \left\{g(v^1), g(v^3)\right\} = \left\{g(v^2)+1, g(v^2)+2\right\}.$$

A lozenge (v^1, v^2, v^3, v^4) for which $(g(v^1), g(v^2), g(v^3), g(v^4)) = (a, a+1, a+2, a+1)$ for some $a \in \{0,1,2\}$ is called *rigid*. The *support* of a regular labeling $g : R_{d,n} \to \mathbb{Z}_3$ is the subset $Supp(g) \subset R_{d,n}$ of vertices of $R_{d,n}$ which are not a vertex v_4 of a rigid lozenge (v^1, v^2, v^3, v^4).

An example of regular labeling is shown in Figure 6.4. By the boundary condition of a regular labeling, any vertex v_4 of a rigid lozenge of g can not be on the boundary of $R_{d,n}$, so that the latter is always contained in $Supp(g)$.

Remark 6.1.2. Although given above in a compact form, there may be better ways of considering a regular labeling for growing n. By seeing $R_{d,n}$ through its hexagonal representation, a regular embedding is equivalent to a tiling of $R_{d,n}$ with either blue or red equilateral triangles of size 1 or lozenges of size 1 with alternating colors on its boundaries, such that the six boundary edges of $R_{d,n}$ are alternatively colored red and blue, starting with the color red on the south edge $\{(v_1, v_2) \in R_{d,n},\ v_2 = 0\}$. The bijection from the former representation to the latter is given by assigning the red (resp. blue) color to any edge of the form $(v, v + e^{2i\pi\ell})$, $0 \leqslant \ell \leqslant 2$ along which the labels of g increase by 1 (resp. decrease by 1), see Figure 6.4 for an example with $n = 4, d = 1$ and Proposition 6.6.9 for a proof of this fact.

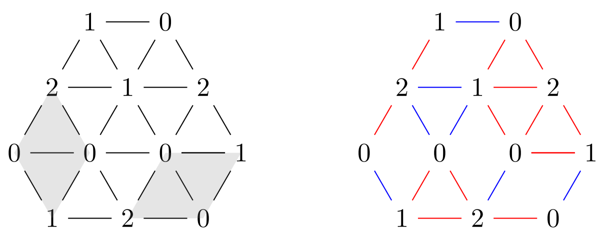

Figure 6.4: A regular labeling on $R_{d,n}$ and its colored representation. Rigid lozenges are shaded.

Definition 6.1.3 (Toric hive cone)**.** A function $f : R_{d,n} \to \mathbb{R}$ is called *toric rhombus concave* with respect to a regular labeling $g : R_{d,n} \to \mathbb{Z}_3$ when $f(v_2) + f(v_4) \geqslant f(v_1) + f(v_3)$ on any lozenge $\ell = (v^1, v^2, v^3, v^4)$, with equality if ℓ is rigid with respect to g.

For any regular labeling g, the *toric hive cone* $\mathcal{C}_g$ with respect to g is the cone

$$\mathcal{C}_g = \left\{ f_{|Supp(g)} \mid f : R_{d,n} \to \mathbb{R} \text{ toric rhombus concave with respect to } g \right\}.$$

As we will see later, for any regular labeling g, $Supp(g)$ has cardinal $(n-1)(n-2)/2 + 3n$ and so is the dimension of $\mathcal{C}_g$. As such, we recover the usual dimension of the classical hive cone from [KT99]. The latter is then a particular case of toric hive cone for $d = 0$. An example of a toric rhombus concave function in the case $n = 3, d = 1$ is given in Figure 6.5.

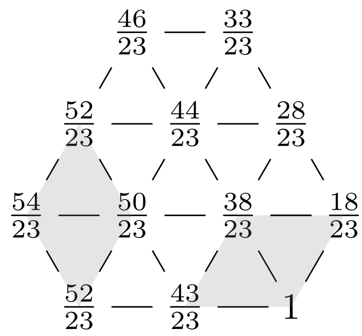

Figure 6.5: A toric rhombus convave function for $n = 3, d = 1$: shaded lozenge are the rigid ones yielding the equality cases in the toric rhombus concavity.

Definition 6.1.4 (Polytope $P^g_{\alpha,\beta,\gamma}$)**.** Let $n \geqslant 3$ and let $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ be such that $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ with $d \in \mathbb{N}$. Let g be a regular labeling on $R_{d,n}$. Then, $P^g_{\alpha,\beta,\gamma}$ is the polytope of $\mathbb{R}^{Supp(g)\setminus\partial R_{d,n}}$ consisting of functions in $\mathcal{C}_g$ such that

$$f^A = \left(\sum_{s=1}^n \beta_s + \sum_{s=1}^i \alpha_s\right)_{0\leqslant i\leqslant n}, \; f^B = \left((d-i)^+ + \sum_{s=1}^i \beta_s\right)_{0\leqslant i\leqslant n}, \; f^C = \left(d + \sum_{s=1}^i \gamma_s\right)_{0\leqslant i\leqslant n}.$$

An example of an element of $P^g_{\alpha,\beta,\gamma}$ for $n = 3$ and $d = 1$ is depicted in Figure 6.5, for $\alpha = \left(\frac{13}{23} \geqslant \frac{6}{23} \geqslant \frac{2}{23}\right)$, $\beta = \left(\frac{18}{23} \geqslant \frac{10}{23} \geqslant \frac{5}{23}\right)$ and $\gamma = \left(\frac{20}{23} \geqslant \frac{9}{23} \geqslant \frac{2}{23}\right)$.

Our main result gives then a formula for the density of the convolution of regular conjugacy classes as a sum of volumes of polytopes coming from $\mathcal{C}_g$ for regular labeling g.

Theorem 6.1.5 (Probability density for product of conjugacy classes)**.** *Let $n \geqslant 3$ and let $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ be such that $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ with $d \in \mathbb{N}$. Then,*

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{(2\pi)^{(n-1)(n-2)/2} \prod_{k=1}^{n-1} k! \Delta'(\mathrm{e}^{2i\pi\gamma})}{n!\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})} \sum_{g:R_{d,n}\to\mathbb{Z}_3 \text{ regular}} \mathrm{Vol}_g\left(P^g_{\alpha,\beta,\gamma}\right), \tag{6.1.1}$$

where $\Delta'(e^{2i\pi\theta}) = 2^{n(n-1)/2} \prod_{i<j} \sin(\pi(\theta_i - \theta_j))$ for $\theta \in \mathcal{H}$ and Vol_g denotes the volume with respect to the Lebesgue measure on $\mathbb{R}^{Supp(g)\setminus\partial R_{d,n}}$.

Note that the case $n = 2$ admits explicit formulas which do not need such a machinery. Numerical experiments for $n = 3$ suggest that there are $\alpha, \beta \in \mathcal{H}_{reg}$ for which any regular labeling g yields a non-empty polytope $P^g_{\alpha,\beta,\gamma}$ for some $\gamma \in \mathcal{H}_{reg}$. However, for a fixed triple $(\alpha, \beta, \gamma) \in \mathcal{H}_{reg}$ there seems to be generically only a strict subset of $\{P^g_{\alpha,\beta,\gamma}\}_{g \text{ regular}}$ which are not empty and do contribute. Finally, remark that we only considered the case of regular conjugacy classes to ensure the existence of a density for the convolution product. Such a hypothesis is regularly assumed, see for example [Wit91; MW99]. However, we expect similar results to hold in cases where less than $n/2$ coordinates of $\mathcal{H}$ are equal, in which case the convolution product is still expected to have a density.

The main application of the previous result is the computation of the volume of moduli spaces of flat SU(n)-connections on the three-punctured sphere. Computing such volume is an important task in the study of the Yang-Mills measure on Riemann surfaces in the small surface limit [For93], and it has been shown in [Wit92; MW99] that this computation for arbitrary compact Riemann surfaces can be reduced to the case of the three-punctured sphere by a sewing phenomenon. A similar inductive procedure is used in [Mir07] to reduce the volume problem for the moduli space of curves to the genus zero case.

Let us denote by Σ_0^3 the sphere with three generic marked points a, b, c removed. We then denote by $\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)$ the moduli space of flat $\mathrm{SU}(n)$-valued connections on Σ_0^3 for which the holonomies around a, b, c respectively belong to $\mathcal{O}_\alpha, \mathcal{O}_\beta$ and $\mathcal{O}_\gamma$. In the specific case of the punctured sphere, this moduli space can be alternatively described as

$$\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma) = \{(U_1, U_2, U_3) \in \mathcal{O}_\alpha \times \mathcal{O}_\beta \times \mathcal{O}_\gamma,\, U_1U_2U_3 = Id_{\mathrm{SU}(n)}\}/\mathrm{SU}(n),$$

where $\mathrm{SU}(n)$ acts diagonally by conjugation, see Section 3.2.5. As a corollary of Theorem 6.1.5, we thus get an expression of the volume of $\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)$ as a sum of volumes of explicit polytopes.

Corollary 6.1.6 (Volume of flat $\mathrm{SU}(n)$-connections on the sphere)**.** *Let $n \geqslant 3$ and consider the canonical volume form on $\mathrm{SU}(n)$. For $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ such that $|\alpha|_1,\ |\beta|_1,\ |\gamma|_1 \in \mathbb{N}$, then $\mathrm{Vol}\left[\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)\right] \neq 0$ only if $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i + \sum_{i=1}^n \gamma_i = n + d$ for some $d \in \mathbb{N}$, in which case*

$$\mathrm{Vol}\left[\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)\right] = \frac{2^{(n+1)[2]}(2\pi)^{(n-1)(n-2)}}{n!\Delta'(\mathrm{e}^{2i\pi\gamma})\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})} \sum_{g: R_{d,n} \to \mathbb{Z}_3 \ \textit{regular}} \mathrm{Vol}_g\left(P^g_{\alpha,\beta,\widetilde{\gamma}}\right),$$

where $\widetilde{\gamma} = (1 - \gamma_n, \dots, 1 - \gamma_1)$ and the polytopes $P^g_{\alpha,\beta,\widetilde{\gamma}}$ are defined in Definition 6.1.4.

Note that the choice of normalization for the volume of $\mathrm{SU}(n)$ slightly differs from the one used in [Wit91] for numerical applications. As a consequence of this corollary, the volume is a piecewise polynomial in α, β, γ, up to the normalization factor coming from the volume of the conjugacy classes. Such a phenomenon, which is a reflect of the underlying symplectic structure, had already been observed in [MW99]. A same phenomenon occurs in the co-adjoint case, see [CZ18; ER18] and in the study of moduli spaces of curves, [Mir07].

6.2 Density formula via the quantum cohomology of the Grassmannians

Let $n \geqslant 1$ and consider $\alpha, \beta \in \mathcal{H}^2_{reg} : \alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ where

$$1 \geqslant \alpha_1 > \alpha_2 > \dots > \alpha_n \geqslant 0 \text{ and } 1 \geqslant \beta_1 > \beta_2 > \dots > \beta_n \geqslant 0.$$

Up to multiplication by the center of $\mathrm{U}(n)$, suppose furthermore without loss of generality that

$$\sum_{i=1}^n \alpha_i = k \text{ and } \sum_{i=1}^n \beta_i = k' \tag{6.2.1}$$

for some $k, k' \in \mathbb{Z}$. Let $A = U\mathrm{e}^{2i\pi\alpha}U^*, B = V\mathrm{e}^{2i\pi\beta}V^*$, where U, V are independent Haar distributed matrices on $\mathrm{U}(n)$. Remark that A and B are respectively uniformly distributed on the conjugacy classes $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$, which lie in $\mathrm{SU}(n) \subset \mathrm{U}(n)$.

The goal of this section, see Theorem 6.2.8, is to give a simple proof of the density formula

(6.2.26) linking the probability $\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta]$ that $AB \in \mathcal{O}(\gamma)$ for $\gamma \in \mathcal{H}$ to the structure constants of the quantum cohomology of Grassmannians defined in Section 6.2.2. Such a semi-classical convergence had been already suggested and proven several times in different forms (see [Wit91] for a similar approach with fusion coefficient and [Man16] for a convergence in distribution). In Section 6.2.1, we recall in Proposition 6.2.1 a classical expression of the density in terms of characters of irreducible representations of $\mathrm{SU}(n)$. In Section 6.2.2, we link the density of Proposition 6.2.1 to the structure constants and derive Theorem 6.2.8.

6.2.1 A first density formula

This part aims at recalling a proof of the formula (6.2.3) which gives the value of $\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta]$ as an infinite sum of characters. A similar treatment of the convolution of orbit measures in the general context of Lie algebras can be found in [Man16, Sec. 7].

Let us denote by $\mathrm{d}g$ the *normalized* Haar measure on $\mathrm{U}(n)$ and for $\theta \in \mathcal{H}$, φ_θ the map

$$\begin{aligned} \varphi_\theta : \mathrm{U}(n) &\to \mathcal{O}(\theta) \subset \mathrm{SU}(n) \\ U &\mapsto U\mathrm{e}^{2i\pi\theta}U^*. \end{aligned}$$

Let us write

$$m_\theta = \varphi_\theta \# \mathrm{d}g \tag{6.2.2}$$

for the push-forward of $\mathrm{d}g$ by φ_θ. The measure m_θ is a measure on $\mathcal{O}(\theta)$ called the orbital measure. For any function $f : \mathcal{O}(\theta) \to \mathbb{R}$,

$$\int_{\mathcal{O}(\theta)} f \mathrm{d}m_\theta = \int_{\mathrm{U}(n)} f(g\mathrm{e}^{2i\pi\theta}g^{-1})dg.$$

Recall that the irreducible representations of the compact group $\mathrm{SU}(n)$ are parametrized by $\lambda \in \mathbb{Z}_{\geqslant 0}^{n-1}$ and we denote by $(\rho_\lambda, V_\lambda)$ the corresponding representation where ρ_λ : $\mathrm{SU}(n) \to V_\lambda$ and $\chi_\lambda : End_{V_\lambda} \to \mathbb{C}, x \mapsto \mathrm{Tr}[x]$ is the associated character.

Proposition 6.2.1 (Induced density of eigenvalues)**.** *Let $(\alpha,\beta) \in \mathcal{H}_{reg}^2$ and let $A, B \in \mathcal{O}(\alpha) \times \mathcal{O}(\beta)$ be two independent random variables sampled from m_α and m_β respectively. Let $C = AB \in \mathcal{O}(\gamma)$ for some random $\gamma \in (\mathbb{R}^n/\mathbb{Z}^n)/S_n$. The density of $\gamma = \gamma_1 > \cdots > \gamma_n \geqslant 0$ is given by the absolute convergent series*

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{|\Delta(\mathrm{e}^{2i\pi\gamma})|^2}{(2\pi)^{n-1}n!} \sum_{\lambda \in \mathbb{Z}_{\geqslant 0}^{n-1}} \frac{1}{\dim V_\lambda} \chi_\lambda\left(\mathrm{e}^{2i\pi\alpha}\right)\chi_\lambda\left(\mathrm{e}^{2i\pi\beta}\right)\chi_\lambda\left(\mathrm{e}^{-2i\pi\gamma}\right). \tag{6.2.3}$$

Another expression of the density (6.2.3) is given in (6.2.24). The rest of this section is devoted to the proof of Proposition 6.2.1.

Definition 6.2.2 (Fourier Transform on $\mathrm{SU}(n)$)**.** Let m be a measure on $\mathrm{SU}(n)$. The *Fourier transform* $\widehat{m}$ of m is defined as

$$\widehat{m} : \lambda \in \mathbb{Z}_{\geqslant 0}^{n-1} \mapsto \widehat{m}(\lambda) := \int_{\mathrm{SU}(n)} \rho_\lambda(g)\mathrm{d}m(g) \in \mathrm{End}_{V_\lambda}. \tag{6.2.4}$$

In the case where $m = m_\theta$, the expression $\widehat{m_\theta}(\lambda)$ is also known as the spherical transform introduced in [ZKF21, eq. (56)].

Lemma 6.2.3 (Fourier Transform of m_θ)**.** *One has, for $\lambda \in \mathbb{Z}_{\geqslant 0}^{n-1}$,*

$$\widehat{m_\theta}(\lambda) = \frac{\chi_\lambda\left(\mathrm{e}^{2i\pi\theta}\right)}{\dim V_\lambda} \mathrm{id}_{V_\lambda}, \tag{6.2.5}$$

where id_{V_λ} is the identity element of V_λ.

Proof. For any $\lambda \in \mathbb{Z}_{\geqslant 0}^{n-1}$ and $g \in \mathrm{SU}(n)$, since the Haar measure is invariant by translation,

$$\begin{aligned}
\widehat{m_\theta}(\lambda)\rho_\lambda(g) &= \left(\int_{\mathrm{U}(n)} \rho_\lambda(h\mathrm{e}^{2i\pi\theta}h^{-1})dh\right)\rho_\lambda(g) \\
&= \int_{\mathrm{U}(n)} \rho_\lambda(h\mathrm{e}^{2i\pi\theta}h^{-1}g)dh \\
&= \rho_\lambda(g)\int_{\mathrm{U}(n)} \rho_\lambda(g^{-1}h\mathrm{e}^{2i\pi\theta}h^{-1}g)dh \\
&= \rho_\lambda(g)\int_{\mathrm{U}(n)} \rho_\lambda(h\mathrm{e}^{2i\pi\theta}h^{-1})dh = \rho_\lambda(g)\widehat{m_\theta}(\lambda).
\end{aligned}$$

Hence, $\widehat{m_\theta}(\lambda)$ is a morphism of the irreducible representation ρ_λ and thus $\widehat{m_\theta}(\lambda) = c \cdot \mathrm{id}_{V_\lambda}$ for some $c \in \mathbb{C}$. One computes the value of c by taking the trace which gives

$$c = \frac{\chi_\lambda(\mathrm{e}^{2i\pi\theta})}{\dim V_\lambda}. \tag{6.2.6}$$

□

Definition 6.2.4 (Convolution of measures)**.** Let m, m' be two measures on $\mathrm{SU}(n)$. Let $m \otimes m'$ be the product measure on $\mathrm{SU}(n) \times \mathrm{SU}(n)$. Define $\mathit{mult} : \mathrm{SU}(n) \times \mathrm{SU}(n) \to \mathrm{SU}(n)$ to be the multiplication on $\mathrm{SU}(n)$: $\mathit{mult}(g_1, g_2) = g_1 g_2$. The *convolution* of m and m', denoted by $m * m'$, is defined as

$$m * m' := \mathit{mult}\#(m \otimes m'), \tag{6.2.7}$$

which means that for any function f on $\mathrm{SU}(n)$,

$$\int_{\mathrm{SU}(n)} f(g)\mathrm{d}(m * m')(g) = \int_{\mathrm{SU}(n)}\int_{\mathrm{SU}(n)} f(g_1 g_2)\mathrm{d}m(g_1)\mathrm{d}m(g_2).$$

For $(\alpha, \beta) \in \mathcal{H}^2$, we write $m_{\alpha,\beta} := m_\alpha * m_\beta$ the convolution of m_α and m_β.

By Definition 6.2.4, the measure $m_{\alpha,\beta}$ is the law on $\mathrm{SU}(n)$ of $C = A \cdot B$ where A and B are sampled from measures μ_α and μ_β on $\mathcal{O}(\alpha)$ and $\mathcal{O}(\beta)$ respectively. Recall that for two measures m, m' on $\mathrm{SU}(n)$,

$$\widehat{m * m'}(\lambda) = \widehat{m}(\lambda)\widehat{m}'(\lambda). \tag{6.2.8}$$

In particular

$$\widehat{m}_{\alpha,\beta}(\lambda) = \widehat{m}_\alpha(\lambda)\widehat{m}_\beta(\lambda). \tag{6.2.9}$$

Recall that we are interested in the measure $\mu_{\alpha,\beta}$. By (6.2.9), one knows how to compute its Fourier transform. Let us define the inverse Fourier transform.

Definition 6.2.5 (Inverse Fourier Transform)**.** Let $f : \lambda \in \mathbb{Z}^{n-1}_{\geqslant 0} \mapsto f(\lambda) \in End_{V_\lambda}$ be a function such that

$$\|f\|^2 = \sum_{\lambda \in \mathbb{Z}^{n-1}_{\geqslant 0}} \dim V_\lambda \cdot \mathrm{Tr}[f(\lambda) f^*(\lambda)] < \infty. \tag{6.2.10}$$

The *inverse Fourier transform* of f is

$$f^\vee : \mathrm{SU}(n) \to \mathbb{C} \tag{6.2.11}$$

$$g \mapsto \sum_{\lambda \in \mathbb{Z}^{n-1}_{\geqslant 0}} \dim V_\lambda \cdot \mathrm{Tr}\left[\rho_\lambda(g^{-1}) f(\lambda)\right]. \tag{6.2.12}$$

In order to apply inverse Fourier transform to $\widehat{m}_{\alpha,\beta}$, one needs to check condition (6.2.10). This is the purpose of the next lemma.

Lemma 6.2.6 (Product Fourier transform is L^2)**.** *For* $(\alpha, \beta) \in \mathcal{H}^2_{reg}$,

$$\sum_{\lambda \in \mathbb{Z}^{n-1}_{\geqslant 0}} \dim(V_\lambda) \,\mathrm{Tr}[\widehat{m}_{\alpha,\beta}(\lambda) \widehat{m}_{\alpha,\beta}(\lambda)^*] < \infty. \tag{6.2.13}$$

Proof. Using (6.2.9) together with (6.2.5), one has

$$\sum_{\lambda \in \mathbb{Z}^{n-1}_{\geqslant 0}} \dim(V_\lambda) \,\mathrm{Tr}[\widehat{m}_{\alpha,\beta}(\lambda) \widehat{m}_{\alpha,\beta}(\lambda)^*] = \sum_{\lambda \in \mathbb{Z}^{n-1}_{\geqslant 0}} \frac{1}{\dim(V_\lambda)^2} \left|\chi_\lambda(\mathrm{e}^{2i\pi\alpha})\right|^2 \left|\chi_\lambda(\mathrm{e}^{2i\pi\beta})\right|^2. \tag{6.2.14}$$

Using Weyl's character formula [CZ18, eq. (21)],

$$\chi_\lambda(\mathrm{e}^{i\theta}) = \frac{\det[\mathrm{e}^{i\theta_r \lambda'_s}]_{1 \leqslant r,s \leqslant n}}{\Delta(\mathrm{e}^{i\theta})} \tag{6.2.15}$$

where $\lambda' = (\lambda_1, \ldots, \lambda_{n-1}, 0) + \rho$ with $\rho = (n-1, \ldots, 0)$ and where $\Delta(x_1, \ldots, x_n) = \prod_{1 \leqslant i < j \leqslant n}(x_i - x_j)$ is the Vandermonde determinant. Recall that by assumption, $\alpha_i \neq \alpha_j$ for $i \neq j$ and the same holds for β, so that the expressions $\Delta(\mathrm{e}^{2i\pi\alpha})$ and $\Delta(\mathrm{e}^{2i\pi\beta})$ are well-defined. The previous sum becomes

$$\frac{1}{|\Delta(\mathrm{e}^{2i\pi\alpha})\Delta(\mathrm{e}^{2i\pi\beta})|^2} \sum_{\lambda_1 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \lambda_n = 0} \frac{\left|\det[\mathrm{e}^{2i\pi\alpha_r \lambda'_s}]\right|^2 \left|\det[\mathrm{e}^{2i\pi\beta_r \lambda'_s}]\right|^2}{\dim(V_\lambda)^2} \tag{6.2.16}$$

$$\leqslant \frac{n^{2n}}{|\Delta(\mathrm{e}^{2i\pi\alpha})\Delta(\mathrm{e}^{2i\pi\beta})|^2} \sum_{\lambda_1 \geqslant \cdots \geqslant \lambda_{n-1} \geqslant \lambda_n = 0} \frac{1}{\dim(V_\lambda)^2} \tag{6.2.17}$$

where we used Hadamard's inequality for the upper bound on determinants. From the identity

$$\dim(V_\lambda) = \frac{\Delta(\lambda')}{sf(n-1)}, \tag{6.2.18}$$

which can be found in [Far08, Cor. 11.2.5] and where $sf(n) = \prod_{1 \leqslant j \leqslant n} j!$, it suffices to show that

$$V_n = \sum_{\lambda_1 > \cdots > \lambda_n = 0} \frac{1}{\Delta(\lambda)^2} \tag{6.2.19}$$

converges for $n \geqslant 2$. One has that $V_2 = \sum_{k\geqslant 1} k^{-2} < \infty$. Let us write

$$\begin{aligned}\sum_{\lambda_1>\dots>\lambda_n=0} \frac{1}{\Delta(\lambda)^2} &= \sum_{\lambda_1>\dots>\lambda_n=0} \prod_{1\leqslant i<j\leqslant n} (\lambda_i-\lambda_j)^{-2} \\ &= \sum_{\lambda_2>\dots>\lambda_n=0} \left[\sum_{\lambda_1>\lambda_2} \prod_{2\leqslant j\leqslant n} (\lambda_1-\lambda_j)^{-2}\right] \prod_{2\leqslant i<j\leqslant n} (\lambda_i-\lambda_j)^{-2} \\ &\leqslant \sum_{\lambda_2>\dots>\lambda_n=0} \left[\sum_{\lambda_1>\lambda_2} (\lambda_1-\lambda_2)^{-2(n-1)}\right] \prod_{2\leqslant i<j\leqslant n} (\lambda_i-\lambda_j)^{-2}\end{aligned}$$

the innermost sum is bounded by $\sum_{k\geqslant 1} k^{-2(n-1)} \leqslant \sum_{k\geqslant 1} k^{-2} = V_2$ for $n \geqslant 2$. Thus,

$$V_n \leqslant V_2 V_{n-1}$$

so that for $n \geqslant 2$, $V_n \leqslant (V_2)^{n-1}$ which proves the convergence.

□

Lemma 6.2.6 shows that the Fourier transform of $\mu_\alpha * \mu_\beta$ is in L^2, so that one can take its inverse Fourier Transform. This leads to the following result.

Lemma 6.2.7 (Inverse Fourier of Convolution)**.** *Let* $(\alpha, \beta) \in \mathcal{H}^2_{reg}$ *and* $g \in \mathrm{SU}(n)$*. Then,*

$$(\widehat{m}_{\alpha,\beta})^\vee(g) = \sum_{\lambda\in\mathbb{Z}^{n-1}_{\geqslant 0}} \frac{1}{\dim V_\lambda} \chi_\lambda(\mathrm{e}^{2i\pi\alpha})\chi_\lambda(\mathrm{e}^{2i\pi\beta})\chi_\lambda(g^{-1}), \tag{6.2.20}$$

where the sum converges in $L^2(\mathrm{SU}(n))$*.*

Proof. Using (6.2.12) together with (6.2.9) yields

$$\begin{aligned}(\widehat{m}_{\alpha,\beta})^\vee(g) &= \sum_{\lambda\in\mathbb{Z}^{n-1}_{\geqslant 0}} \dim V_\lambda \cdot \mathrm{Tr}[\rho_\lambda(g^{-1})\widehat{m}_\alpha(\lambda)\widehat{m}_\beta(\lambda)] && (6.2.21)\\ &= \sum_{\lambda\in\mathbb{Z}^{n-1}_{\geqslant 0}} \dim V_\lambda \cdot \mathrm{Tr}\left[\frac{\chi_\lambda(\mathrm{e}^{2i\pi\alpha})\chi_\lambda(\mathrm{e}^{2i\pi\beta})}{\dim V_\lambda^2}\rho_\lambda(g^{-1})\right] && (6.2.22)\\ &= \sum_{\lambda\in\mathbb{Z}^{n-1}_{\geqslant 0}} \frac{1}{\dim V_\lambda}\chi_\lambda(\mathrm{e}^{2i\pi\alpha})\chi_\lambda(\mathrm{e}^{2i\pi\beta})\chi_\lambda(g^{-1}). && (6.2.23)\end{aligned}$$

□

Proof of Proposition 6.2.1. The induced density on γ is given by the density (6.2.20) multiplied by the Jacobian of the diagonalization map $g \mapsto V\mathrm{e}^{2i\pi\gamma}V^*$ with $\gamma = (\gamma_1, \dots, \gamma_n)$ such that $\sum \gamma_i \in \mathbb{Z}$. Since this Jacobian is $\frac{|\Delta(\mathrm{e}^{2i\pi\gamma})|^2}{(2\pi)^{n-1}n!}$, see [Far08, Thm 11.2.1], we obtain the desired expression. □

Writing the density (6.2.3) using (6.2.15) and the fact that $\dim V_\lambda = \frac{\Delta(\lambda')}{sf(n-1)}$ yields

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{\Delta(\mathrm{e}^{2i\pi\gamma})sf(n-1)}{(2\pi)^{n-1}n!\Delta(\mathrm{e}^{2i\pi\alpha})\Delta(\mathrm{e}^{2i\pi\beta})} \sum_{\lambda\in\mathbb{Z}^{n-1}_{\geq 0}} \frac{\det\left[\mathrm{e}^{2i\pi\alpha_r\lambda'_s}\right]\det\left[\mathrm{e}^{2i\pi\beta_r\lambda'_s}\right]\det\left[\mathrm{e}^{-2i\pi\gamma_r\lambda'_s}\right]}{\Delta(\lambda')}.$$

Moreover, for $\theta = (\theta_1 \geqslant \cdots \geqslant \theta_n)$,

$$\begin{aligned}\Delta(\mathrm{e}^{2i\pi\theta}) &= \prod_{1\leqslant r<s\leqslant n} \left(\mathrm{e}^{2i\pi\theta_r} - \mathrm{e}^{2i\pi\theta_s}\right)\\ &= \mathrm{e}^{i\pi\sum_{1\leqslant r<s\leqslant n}(\theta_r+\theta_s)} \prod_{1\leqslant r<s\leqslant n} \left[2i\sin\left(\pi(\theta_r-\theta_s)\right)\right]\end{aligned}$$

Let us compute

$$\begin{aligned}\sum_{1\leqslant r<s\leqslant n}(\theta_r+\theta_s) &= \sum_{r=1}^{n-1}\sum_{s=r+1}^{n}(\theta_r+\theta_s)\\ &= \sum_{r=1}^{n-1}\left((n-r)\theta_r + \sum_{s=r+1}^{n}\theta_s\right)\\ &= \sum_{r=1}^{n-1}(n-r)\theta_r + \sum_{r=1}^{n}(r-1)\theta_r\\ &= (n-1)\sum_{r=1}^{n}\theta_r.\end{aligned}$$

Thus, if one sets $\Delta'(e^{2i\pi\theta}) = 2^{n(n-1)/2}\prod_{i<j}\sin\left(\pi(\theta_i-\theta_j)\right)$,

$$\begin{aligned}\frac{\Delta(\mathrm{e}^{2i\pi\gamma})}{\Delta(\mathrm{e}^{2i\pi\alpha})\Delta(\mathrm{e}^{2i\pi\beta})} &= \mathrm{e}^{i\pi(n-1)(|\gamma|-|\alpha|-|\beta|)} i^{-n(n-1)/2}\frac{\Delta'(\mathrm{e}^{2i\pi\gamma})}{\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})}\\ &= (-1)^{d(n-1)} i^{-n(n-1)/2}\frac{\Delta'(\mathrm{e}^{2i\pi\gamma})}{\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})}\end{aligned}$$

where for the last equation, we used that $|\alpha| + |\beta| = |\gamma| + d$. Therefore,

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{sf(n-1)(2\pi)^{(n-1)(n-2)/2}\Delta'(\mathrm{e}^{2i\pi\gamma})}{\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})n!}J[\gamma|\alpha,\beta], \tag{6.2.24}$$

where $\Delta'(\mathrm{e}^{2i\pi\theta})$ has been defined in Theorem 6.1.5 and where

$$J[\gamma|\alpha,\beta] = \frac{(-1)^{d(n-1)}}{(2i\pi)^{n(n-1)/2}}\sum_{\lambda\in\mathbb{Z}^{n-1}_{\geq 0}}\frac{1}{\Delta(\lambda')}\det\left[\mathrm{e}^{2i\pi\alpha_r\lambda'_s}\right]\det\left[\mathrm{e}^{2i\pi\beta_r\lambda'_s}\right]\det\left[\mathrm{e}^{-2i\pi\gamma_r\lambda'_s}\right]. \tag{6.2.25}$$

is called the volume function for the unitary Horn problem.

6.2.2 Link with quantum cohomology of the Grassmannians

The goal of this section is to link the volume function (6.2.25) with structure constants of the quantum cohomology ring of Grassmannians $QH^*(\mathbb{G}r)$ in the same way as the volume function in the coadjoint case is related to the classical cohomology ring of Grassmannians, see [CZ18]. The structure constants in the unitary case are the Gromov–Witten invariants, which are related to characters via [Rie01, Cor. 6.2]. We refer the reader to [MS04], [Buc03] and Section 3.2.3 for an introduction to the subject.

For $N \geqslant n$, let us denote by $\mathbb{Z}^n_{N-n}$ the set of partition $\lambda \in \mathbb{Z}^n$ such that $N-n \geqslant \lambda_1 \geqslant \cdots \geqslant$

$\lambda_n \geqslant 0$. Then, the ring $QH^*(\mathbb{G}r(n,N))$ has an additive basis $(q^d \otimes \sigma_\lambda, d \geqslant 0, \lambda \in \mathbb{Z}^n_{N-n})$. We will denote by $c^{\nu,d}_{\lambda,\mu}$ the structure constants of this ring so that

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu,d\geqslant 0} c^{\nu,d}_{\lambda,\mu} q^d \otimes \sigma_\nu .$$

where the sum is over pairs $(\nu,d) \in \mathbb{Z}^n_{N-n} \times \mathbb{N}$ such that $|\lambda|+|\mu| = |\nu| + Nd$. The structure coefficients $c^{\nu,d}_{\lambda,\mu}$ are the degree d Gromov–Witten invariants associated to the Schubert cycles $\sigma_\lambda, \sigma_\mu, \sigma_{\nu^\vee}$, see [Rie01, Cor. 6.2]. The main result of this section is Theorem 6.2.8 below.

Theorem 6.2.8 (Density as limit of quantum coefficients)**.** *Let $(\alpha,\beta,\gamma) \in \mathcal{H}^3_{reg}$. For each $N \geqslant 1$, let $(\lambda_N, \mu_N, \nu_N)$ be three partitions in $\mathbb{Z}^n_{N-n}$ such that $|\lambda_N| + |\mu_N| = |\nu_N| + dN$ for some $d \in \mathbb{Z}_{\geqslant 0}$ and such that $\frac{1}{N}\lambda_N = \alpha + o(1)$, $\frac{1}{N}\mu_N = \beta + o(1)$ and $\frac{1}{N}\nu_N = \gamma + o(1)$ as $N \to +\infty$. Then,*

$$\lim_{N\to\infty} N^{-(n-1)(n-2)/2} c^{\nu_N,d}_{\lambda_N,\mu_N} = J[\gamma|\alpha,\beta] = \frac{\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})n!}{sf(n-1)(2\pi)^{(n-1)(n-2)/2}\Delta'(\mathrm{e}^{2i\pi\gamma})} \mathrm{d}\mathbb{P}[\gamma|\alpha,\beta]. \tag{6.2.26}$$

The rest of this section is devoted to the proof of Theorem 6.2.8. In subsection 6.2.3 we prove a determinantal formula for the coefficients $c^{\nu,d}_{\lambda,\mu}$ along with some results on the quantities involved in the expression. In subsection 6.2.4 we prove Theorem 6.2.8 using Lemmas 6.2.16 and 6.2.15.

6.2.3 Determinantal expression for $c^{\nu,d}_{\lambda,\mu}$

For $1 \leqslant n \leqslant N$, set

$$I_{n,N} = \left\{ (I_1,\dots,I_n) \in \left(\mathbb{Z} + \left(\frac{1}{2}\right)^{(n-1)[2]}\right)^n \mid -\frac{n-1}{2} \leqslant I_n < \dots < I_1 \leqslant N - \frac{n+1}{2} \right\}.$$

For $I \in I_{n,N}$, let us introduce the notations $\xi = \exp(2i\pi/N)$ and $\xi^I = (\xi^{I_1}, \dots, \xi^{I_n})$.

Lemma 6.2.9 (Determinantal expression for $c^{\nu,d}_{\lambda,\mu}$)**.** *Let λ,μ,ν such that $|\lambda|+|\mu| = |\nu| + Nd$. Then,*

$$c^{\nu,d}_{\lambda,\mu} = \frac{1}{N^n} \sum_{I \in I_{n,N}} \frac{\det\left[\mathrm{e}^{\frac{2i\pi I_r(\lambda_s+(s-1))}{N}}\right] \det\left[\mathrm{e}^{\frac{2i\pi I_r(\mu_s+(s-1))}{N}}\right] \det\left[\mathrm{e}^{-\frac{2i\pi I_r(\nu_s+(s-1))}{N}}\right]}{\Delta(\xi^I)}. \tag{6.2.27}$$

Proof. Let us denote by

$$S_\lambda(x_1,\dots,x_n) = \frac{\det\left[x_r^{(\lambda_s+(s-1))}, 1 \leqslant r,s \leqslant n\right]}{\Delta(x)}$$

the Schur function corresponding to the partition λ. Using [Rie01, Corollary 6.2]:

$$c^{\nu,d}_{\lambda,\mu} = \frac{1}{N^n} \sum_{I \in I_{n,N}} S_\lambda(\xi^I) S_\mu(\xi^I) S_{\nu^\vee}(\xi^I) \frac{|\Delta(\xi^I)|^2}{S_{(N-n)}(\xi^I)}. \tag{6.2.28}$$

Moreover, by [Rie01, eq. (4.3)], one has

$$\frac{S_{\nu^\vee}(\xi^I)}{S_{(N-n)}(\xi^I)} = \overline{S_\nu(\xi^I)}$$

so that

$$c_{\lambda,\mu}^{\nu,d} = \frac{1}{N^n} \sum_{I \in I_{n,m}} S_\lambda(\xi^I) S_\mu(\xi^I) S_\nu(\overline{\xi^I}) |\Delta(\xi^I)|^2$$
$$= \frac{1}{N^n} \sum_{I \in I_{n,m}} \frac{\det\left[\mathrm{e}^{\frac{2i\pi I_r(\lambda_s+(s-1))}{N}}\right] \det\left[\mathrm{e}^{\frac{2i\pi I_r(\mu_s+(s-1))}{N}}\right] \det\left[\mathrm{e}^{-\frac{2i\pi I_r(\nu_s+(s-1))}{N}}\right]}{\Delta(\xi^I)}. \quad (6.2.29)$$

□

We are interested in the asymptotic behaviour of the previous expression as $N \to \infty$. Let us define

$$F(I, \lambda, \mu, \nu, N) = \frac{\det\left[\mathrm{e}^{\frac{2i\pi I_r(\lambda_s+(s-1))}{N}}\right] \det\left[\mathrm{e}^{\frac{2i\pi I_r(\mu_s+(s-1))}{N}}\right] \det\left[\mathrm{e}^{-\frac{2i\pi I_r(\nu_s+(s-1))}{N}}\right]}{\Delta(\xi^I)}. \quad (6.2.30)$$

Lemma 6.2.10 (Translation invariance)**.** *Let $I \in \left(\frac{1}{2}\mathbb{Z}\right)^n$ and $a \in \frac{1}{2}\mathbb{Z}$. We still assume that $|\lambda| + |\mu| = |\nu| + Nd$ for some $d \in \mathbb{Z}_{\geqslant 0}$. Then,*

$$F(I + a, \lambda, \mu, \nu, N) = (-1)^{2ad} F(I, \lambda, \mu, \nu, N). \quad (6.2.31)$$

Proof. Since

$$\det\left[\exp\left(\frac{2i\pi(I_r + a)(\lambda_s + s - 1)}{N}\right)\right] = \det\left[\exp\left(\frac{2i\pi I_r(\lambda_s + s - 1)}{N}\right)\right] \cdot \exp\left(a\frac{2i\pi(|\lambda| + \sum_{l=0}^{n-1} l)}{N}\right),$$

the numerator of $F(I + a, \lambda, \mu, \nu, N)$ is the one of $F(I, \lambda, \mu, \nu, N))$ times the factor

$$\exp\left(a\frac{2i\pi}{N}\left(|\lambda| + |\mu| - |\nu| + n(n-1)/2\right)\right) = (-1)^{2ad} \exp\left(a\frac{i\pi n(n-1)}{N}\right)$$

since $|\lambda| + |\mu| - |\nu| = dN$. The Vandermonde in the denominator of $F(I + a, \lambda, \mu, \nu, N)$ is

$$\Delta(\xi^{I+a}) = \prod_{1 \leqslant r < s \leqslant n} \left(\exp\left(\frac{2i\pi(I_r + a)}{N}\right) - \exp\left(\frac{2i\pi(I_s + a)}{N}\right)\right)$$
$$= \prod_{1 \leqslant r < s \leqslant n} \left(\exp\left(\frac{2i\pi I_r}{N}\right) - \exp\left(\frac{2i\pi I_s}{N}\right)\right) \exp\left(a\frac{i\pi n(n-1)}{N}\right)$$
$$= \Delta(\xi^I) \exp\left(a\frac{i\pi n(n-1)}{N}\right)$$

so that the quotient cancels the common additional factor appearing in the numerator and denominator of $F(I + a, \lambda, \mu, \nu, N)$. □

From Lemma 6.2.10, we can shift I by $a = \frac{n-1}{2}$ so that

$$F\left(I + \frac{n-1}{2}, \lambda, \mu, \nu, N\right) = (-1)^{d(n-1)} F(I, \lambda, \mu, \nu, N). \quad (6.2.32)$$

In the following, we will assume that $0 \leqslant I_n < \cdots < I_1 \leqslant N - 1$ and that the $I \in \mathbb{Z}^n$. Denote by $J_{n,N}$ the set

$$J_{n,N} = \left\{I \in \{0, \ldots, N-1\}^n,\ I_1 > I_2 > \ldots > I_n\right\}.$$

Definition 6.2.11 (Action Φ_N and orbits)**.** The translation action of $\mathbb{Z}$ on $J_{n,N}$ is given by

$$\begin{aligned}\Phi_N : \mathbb{Z} \times J_{n,N} &\to J_{n,N} \\ (l, I = (I_1 > \cdots > I_n)) &\mapsto I + (l, \ldots, l)\ [N]\end{aligned}$$

where the tuple $I+(l,\ldots,l)[N]$ consists of the sequence of elements $I_1+l[N],\ldots,I_n+l[N]$ sorted in the decreasing order.

Lemma 6.2.10 shows that F is invariant under the action of Φ_N.

In order to give some properties of orbits of Φ_N, let us recall the lexicographic order on $\mathbb{Z}^n_{\geqslant 0}$. For $I, J \in \mathbb{Z}^{n-1}_{\geqslant 0}$, let $r^* = \inf\{1 \leqslant r \leqslant n \mid I_r \neq J_r\}$, with the convention that $r^* = 0$ if $I = J$. We say that $I > J$ if $I_{r^*} > J_{r^*}$ and $I < J$ if $I_{r^*} < J_{r^*}$. This defines a total order on $\mathbb{Z}^n_{\geqslant 0}$ and by restriction on $J_{n,N}$.

Let $Orbits(N)$ denote the orbits of the action of Φ_N on $J_{n,N}$. For $\Omega_N \in Orbits(N)$, denote by $\min(\Omega_N)$ its minimal element with respect to the lexicographic order. Then, necessarily, $(\min(\Omega_N))_n = 0$, otherwise $\Phi(-1, \min(\Omega_N))$ would be an element of Ω_N strictly inferior to $\min(\Omega_N)$. For an ordered n-tuple I of $\{0,\ldots,N-1\}^n$, let $\Omega(I,N)$ denote its orbit under the action of Φ_N.

Lemma 6.2.12 (Orbit structure for large N)**.** *Let $I = (I_1 > \cdots > I_{n-1} > I_n = 0)$. Then, for N large enough, the orbit of I under the action of Φ_N has cardinal N and I is its minimal element:*

$$\exists M = M(I), \forall N \geqslant M : I = \min(\Omega(I,N)) \text{ and } |\Omega(I,N)| = N. \tag{6.2.33}$$

Proof. Let $G_{I,N}$ denote the stabilizer of I under Φ_N. Then, $N\mathbb{Z} \subset G_{I,N}$ so that $G_{I,N} = p_N\mathbb{Z}$ for some $p_N \geqslant 1$ such that $p_N|N$. Set $d(N) = N - I_1 \geqslant 1$. Then, for $p_N\mathbb{Z}$ to be the stabilizer of I, one must have $d(N) \leqslant p_N$ (recall that $I_n = 0$), for otherwise $I_1 + p_N \notin \{I_2,\ldots,I_n\}$. However, as I_1 is fixed, for $N > 2I_1$, $\frac{N}{2} < d(N) \leqslant p_N$ which implies with $p_N|N$ that $p_N = N$. Hence, for such N, the corresponding orbit has cardinal N.

Let us show that I is minimal in its orbit $\Omega(I,N)$ when N is large enough. Take N such that $I_1 < \frac{N}{2}$. The only points J in the orbit of I such that $J_n = 0$ and $J \neq I$ are

$$\{\Phi_N(-I_{n-1}, I), \Phi_N(-I_{n-2}, I), \ldots, \Phi_N(-I_1, I)\}$$

These tuples are all strictly greater than I since $N - (I_k - I_{k+1}) - I_1 \geqslant N - 2I_1 > 0$. Since the minimal element of an orbit must have $I_n = 0$, the only possibility is I. ☐

Lemma 6.2.13 (Orbit decomposition)**.** *Let n, N be fixed. Then,*

$$\sum_{I \in I_{n,N}} F(I,\lambda,\mu,\nu,N) = (-1)^{d(n-1)} \sum_{I : I_n = 0} 1_{I=\min(\Omega(I,N))} |\Omega(I,N)| F(I,\lambda,\mu,\nu,N). \tag{6.2.34}$$

Proof of Lemma 6.2.13. By the translation invariance of Lemma 6.2.10,

$$\sum_{I \in I_{n,N}} F(I,\lambda,\mu,\nu,N) = (-1)^{d(n-1)} \sum_{0 \leqslant I_n < \cdots < I_1 \leqslant N-1} F(I,\lambda,\mu,\nu,N)$$

We decompose the elements $0 \leqslant I_n < \cdots < I_1 \leqslant N-1$ along orbits of the action defined in Section 6.2.3.

$$\begin{aligned}\sum_{0\leqslant I_n<\cdots<I_1\leqslant N-1} F(I,\lambda,\mu,\nu,N) &= \sum_{\Omega\in Orbits(N)} \sum_{I\in\Omega} F(I,\lambda,\mu,\nu,N)\\ &= \sum_{\Omega\in Orbits(N)} |\Omega| F(\min(\Omega),\lambda,\mu,\nu,N)\\ &= \sum_{0\leqslant I_n<\cdots<I_1\leqslant N-1} 1_{I=\min(\Omega(I,N))}|\Omega(I,N)|F(I,\lambda,\mu,\nu,N)\end{aligned}$$

□

We will need the following result which asserts that I_1/N cannot be arbitrary close to one.

Lemma 6.2.14 (Uniform spacing of I_1)**.** *Let n be fixed. Then,*

$$\forall N \geqslant n, \forall\ \Omega \in Orbits(N) : \frac{(\min(\Omega))_1}{N} \leqslant 1 - \frac{1}{n}.$$

Proof of Lemma 6.2.14. Let $N \geqslant n$ and consider $\Omega \in Orbits(N)$. Denote $I = \min(\Omega)$ its minimal element. Assume for the sake of contradiction that $I_1 > N - \frac{N}{n}$. Divide the interval $]0, N]$ in n disjoint sub-intervals $P_1, \ldots P_n$ of length $\frac{N}{n}$ with

$$P_j = \left(j - 1\frac{N}{n}, j\frac{N}{n}\right], 1 \leqslant j \leqslant n.$$

Since $I_1 > N - \frac{N}{n}$ and $I_n = 0$, I_1 and I_n both belong to the last interval P_n. There are $n-2$ remaining elements $I_2 > \cdots > I_{n-1}$ to be placed inside the $n-1$ unused intervals $P_1, \ldots, P_{n-1}$ and P_n. Thus, there exists $1 \leqslant j \leqslant n-1$ such that $I \cap P_j = \emptyset$. Take the maximal such j and consider $r = \max\{l \in [1,n] : I_l \geqslant j\frac{N}{n}\}$ the index of the smallest element of I greater than P_j. We claim that

$$J = \Phi_N(-I_r, I) < I$$

Indeed, since P_j is empty, $J_1 \leqslant N - \frac{N}{n} < I_1$. This contradicts the fact that I is minimal in the orbit Ω_N. See Figure 6.6 for an illustration of the argument.

□

6.2.4 Convergence of scaled coefficients

Lemma 6.2.15 (Control of $F(I,\lambda,\mu,\nu,N)$)**.** *Let n, N be fixed with $n \geqslant 3$. Then,*

$$\frac{N^{-n+1}}{N^{(n-1)(n-2)/2}}|F(I,\lambda,\mu,\nu,N)| \leqslant C_I\ , \tag{6.2.35}$$

for some C_I such that $\sum_{I:I_n=0} C_I < \infty$.

Proof of Lemma 6.2.15. One has

$$|F(I,\lambda,\mu,\nu,N)| = \left|\frac{\det[e^{\frac{2i\pi I_r\lambda'_s}{N}}]\det[e^{\frac{2i\pi I_r\mu'_s}{N}}]\det[e^{-\frac{2i\pi I_r(\nu)'_s}{N}}]}{\Delta(\xi^I)}\right| \leqslant \frac{n^{3n}}{|\Delta(\xi^I)|}.$$

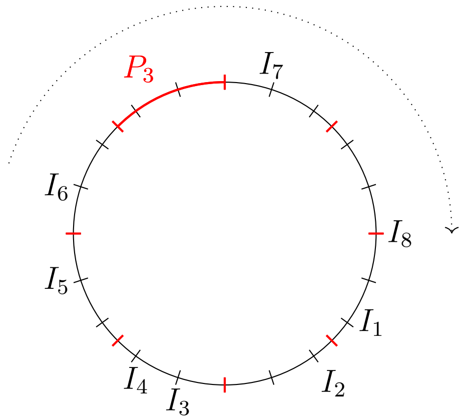

Figure 6.6: Illustration of the argument for $N = 20$ and $n = 8$. Red ticks are the $j\frac{N}{n}$ for $0 \leqslant j \leqslant n-1$ delimiting the P_j's. Here $j = 3$ is the maximal index for which P_j is empty, see the red arc for P_3 and $r = 6$ with $I_r = 9$. The rotation $\Phi_N(-I_r, I) = \Phi_N(-9, I) = J$ is represented by the dotted arrow. J has $J_1 = I_7 - 9[20] = 15$ which is strictly inferior to $I_1 = 18$ leading to a contradiction as I should be minimal in its orbit.

First,

$$\frac{1}{|\Delta(\xi^I)|} = \prod_{1\leqslant r<s\leqslant n} \left|\exp\left(\frac{2i\pi I_r}{N}\right) - \exp\left(\frac{2i\pi I_s}{N}\right)\right|^{-1}$$
$$= \prod_{1\leqslant r<s\leqslant n} \left|2\sin\left(\frac{\pi(I_r - I_s)}{N}\right)\right|^{-1}.$$

Recall that on $[0, c]$ for $0 < c < \pi$, one has by concavity $\sin(x) \geqslant \frac{\sin(c)}{c}x$. Using Lemma 6.2.14 for I minimal in its orbit,

$$\forall 1 \leqslant r < s \leqslant n : \pi\frac{I_r - I_s}{N} \leqslant \pi\frac{I_1}{N} \leqslant \pi\left(1 - \frac{1}{n}\right),$$

so that

$$\sin\left(\frac{\pi(I_r - I_s)}{N}\right) \geqslant c_n\frac{I_r - I_s}{N}$$

with $c_n = \frac{\sin\pi(1-1/n)}{(1-1/n)}$. Thus,

$$\frac{1}{|\Delta(\xi^I)|} \leqslant \left(\frac{N}{2c_n}\right)^{n(n-1)/2} \prod_{1\leqslant r<s\leqslant n} \frac{1}{I_r - I_s}.$$

It remains to prove that $\sum_{I:I_n=0} \frac{1}{\Delta(I)} < \infty$. We will proceed by induction on n. For $n = 3$, the sum is

$$\sum_{I_1>I_2>I_3=0} \frac{1}{(I_1 - I_2)I_1I_2} = \sum_{I_2\geqslant 1} \frac{1}{I_2} \sum_{I_1\geqslant I_2+1} \frac{1}{(I_1 - I_2)I_1}.$$

Moreover, for $I_2 \geqslant 1$,

$$\sum_{I_1\geqslant I_2+1} \frac{1}{(I_1 - I_2)I_1} \leqslant \frac{1}{I_2+1} + \int_{I_2+1}^{\infty} \frac{1}{t(t - I_2)}\mathrm{d}t = \frac{1}{I_2+1} + \frac{\ln(I_2+1)}{I_2}$$

which proves the convergence for $n = 3$ since $\sum_{I_2 \geqslant 1} \frac{1}{I_2} \left(\frac{1}{I_2+1} + \frac{\ln(I_2+1)}{I_2} \right) = C < \infty$. For $n \geqslant 4$,

$$\sum_{I_1 > \cdots > I_{n-1} > I_n = 0} \prod_{1 \leqslant r < s \leqslant n} (I_r - I_s)^{-1} = \sum_{I_2 > \cdots > I_{n-1} > I_n = 0} \prod_{2 \leqslant r < s \leqslant n} (I_r - I_s)^{-1} \sum_{I_1 > I_2} \prod_{2 \leqslant s \leqslant n} (I_1 - I_s)^{-1}$$

and, since

$$\sum_{I_1 > I_2} \prod_{2 \leqslant s \leqslant n} (I_1 - I_s)^{-1} \leqslant \sum_{I_1 > I_2} (I_1 - I_2)^{-(n-1)} \leqslant c_3 = \frac{\pi^2}{6},$$

we have

$$\sum_{I_1 > \cdots > I_{n-1} > I_n = 0} \prod_{1 \leqslant r < s \leqslant n} (I_r - I_s)^{-1} \leqslant c_3 \sum_{I_2 > \cdots > I_{n-1} > I_n = 0} \prod_{2 \leqslant r < s \leqslant n} (I_r - I_s)^{-1} \leqslant c_3^{n-3} C < \infty.$$

Therefore,

$$\frac{N^{-n+1}}{N^{(n-1)(n-2)/2}} |F(I, \lambda, \mu, \nu, N)| \leqslant \frac{n^{3n}}{|\Delta(\xi^I)|} = C_I.$$

with $\sum_{I : I_n = 0} C_I < \infty$ as wanted.

□

Lemma 6.2.16 (Pointwise convergence)**.** *Let $(\alpha, \beta, \gamma) \in \mathcal{H}^3$ such that $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ for $d \in \mathbb{N}$. For $N \geqslant 1$, let $(\lambda_N, \mu_N, \nu_N)$ be three partition in $\mathbb{Z}_{N-n}^n$ such that $|\lambda_N| + |\mu_N| = |\nu_N| + dN$ for some $d \in \mathbb{Z}_{\geqslant 0}$ and such that $\frac{1}{N}\lambda_N = \alpha + o(1)$, $\frac{1}{N}\mu_N = \beta + o(1)$ and $\frac{1}{N}\nu_N = \gamma + o(1)$ as $N \to +\infty$. Let $I = I_1 > \cdots > I_{n-1} > I_n = 0$ be fixed. Then,*

$$\lim_{N \to \infty} \frac{N^{-n} \delta_{I = \min(\Omega(I,N))} |\Omega(I, N)|}{N^{(n-1)(n-2)/2}} F(I, \lambda_N, \mu_N, \nu_N, N) \tag{6.2.36}$$

$$= \lim_{N \to \infty} \frac{N^{-n+1}}{N^{(n-1)(n-2)/2}} F(I, \lambda_N, \mu_N, \nu_N, N) \tag{6.2.37}$$

$$= (2i\pi)^{-n(n-1)/2} \frac{1}{\Delta(I)} \det[e^{2i\pi\alpha_r I_s}] \det[e^{2i\pi\beta_r I_s}] \det[e^{-2i\pi\gamma_r I_s}]. \tag{6.2.38}$$

Proof of Lemma 6.2.16. The first equality is derived from Lemma 6.2.12 which implies that for any I and N large enough

$$\delta_{I = \min(\Omega(I,N))} |\Omega(I, N)| = N.$$

For a fixed $n \geqslant 3$, by continuity,

$$\lim_{N \to \infty} \det \left[\exp \left(\frac{2i\pi I_r (\lambda_{N,s} + s - 1)}{N} \right) \right] = \det[\exp(2i\pi I_r \alpha_s)]$$

$$\lim_{N \to \infty} \det \left[\exp \left(\frac{2i\pi I_r (\mu_{N,s} + s - 1)}{N} \right) \right] = \det[\exp(2i\pi I_r \beta_s)]$$

$$\lim_{N \to \infty} \det \left[\exp \left(-\frac{2i\pi I_r (\nu_{N,s} + s - 1)}{N} \right) \right] = \det[\exp(-2i\pi I_r \gamma_s)]$$

$$\lim_{N \to \infty} \frac{N^{-n+1}}{N^{(n-1)(n-2)/2} \Delta(\xi^I)} = \left(\frac{1}{2i\pi} \right)^{n(n-1)/2} \frac{1}{\Delta(I)}$$

where for the last convergences, we used that $\sin\left(\frac{\pi(I_r - I_s)}{N}\right) \sim \frac{\pi(I_r - I_s)}{N}$ for a fixed subset I. The four convergences above imply the result.

□

Proof of Theorem 6.2.8. From (6.2.29), together with Lemma 6.2.13, one has

$$c_{\lambda_N,\mu_N}^{\nu_N,d} = N^{-n}(-1)^{d(n-1)} \sum_{I:I_n=0} \delta_{I=\min(\Omega(I,N))}|\Omega(I,N)|F(I,\lambda_N,\mu_N,\nu_N,N).$$

By Lemma 6.2.16 and Lemma 6.2.15 using the dominated convergence theorem we have that

$$\begin{aligned}\lim_{N\to\infty} N^{-(n-1)(n-2)/2}c_{\lambda_N,\mu_N}^{\nu_N,d} &= \frac{(-1)^{d(n-1)}}{(2i\pi)^{n(n-1)/2}} \sum_{I:I_n=0} \frac{\det[\mathrm{e}^{2i\pi\alpha_r I_s}]\det[\mathrm{e}^{2i\pi\beta_r I_s}]\det[\mathrm{e}^{-2i\pi\gamma_r I_s}]}{\Delta(I)}\\ &= J[\gamma|\alpha,\beta],\end{aligned}$$

where $J[\gamma|\alpha,\beta]$ was defined in (6.2.25) and is such that

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{sf(n-1)(2\pi)^{(n-1)(n-2)/2}\Delta'(\mathrm{e}^{2i\pi\gamma})}{\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})n!}J[\gamma|\alpha,\beta].$$

□

6.3 Puzzles of the quantum cohomology of Grassmannians and their skeleton

The main goal of this section is a rewriting of the puzzle formula of [Buc+16] for the expression of quantum LR-coefficient in terms of a more compact form approaching the hive model yielding the classical LR-coefficients, see [KT99]. This is done by encoding a puzzle by a graph. We first present the underlying tiling region in Section 6.3.1, which corresponds to the puzzle model, presented in Section 6.3.2. In Section 6.3.3, we represent a puzzle via partitions of edges, vertices and faces of the triangular lattice. In Section 6.3.4, we define the graph of a puzzle using the previous partitions.

6.3.1 Triangular grid

Definition 6.3.1 (Triangular grid)**.** The *triangular grid* of size N, denoted by T_N, is the planar graph whose vertices are the set $V_N = \{r + se^{i\pi/3}, r, s \in \mathbb{N}, r + s \leqslant N\}$ and edges are the set $E_N = \{(x, y), x, y \in T_N, |y - x| = 1\}$.
The set F_N of *faces* of T_N are triangles which are called direct (resp. reversed) if the corresponding vertices $(x^1, x^2, x^3) \in V_N^3$ can be labelled in such a way that $x^2 - x^1 = 1$ and $x_3 - x_1 = e^{i\pi/3}$ (resp. $x_3 - x_1 = e^{-i\pi/3}$).
Any union of two triangles sharing an edge e is called a *lozenge*, and e is then called the *middle edge* of the lozenge.

We denote by F_N^+ (resp. F_N^-) the set of directed (resp. reversed) triangles, so that $F_N = F_N^+ \cup F_N^-$. For $e \in E_N, f \in F_N$, we write $e \in f$ when e is an edge on the boundary of f. Remark that the set of edges can be partitioned into three subset depending on their orientation. If $x = r + se^{i\pi/3} \in T_N$ we define three coordinates

$$x_0 = N - (r + s),\ x_1 = r,\ x_2 = s,$$

and we usually denote an element of T_N by those three coordinates to emphasize the threefold symmetry of the triangle. We say that an edge $e = (x, x + v)$ is of type ℓ, $\ell \in \{0, 1, 2\}$ when $v = e^{i\pi+2\ell i\pi/3}$, see Figure 6.7.

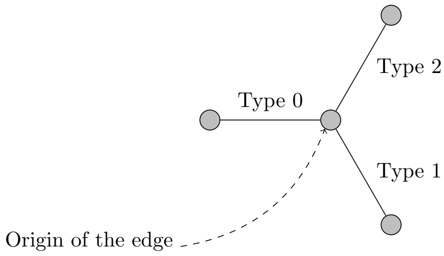

Figure 6.7: Type of an edge in T_N

Definition 6.3.2 (Edge coordinates)**.** For $x \in T_N$ and $\ell \in \{0,1,2\}$ such that $x + e^{i\pi+2\ell i\pi/3} \in T_N$, the *coordinates* of the edge $e = (x, x + e^{i\pi+2\ell i\pi/3})$ of type ℓ is the triple (x_0, x_1, x_2) and we denote by $e_i = x_i$ the i–th coordinate. We define the *height* $h(e)$ of an edge of type ℓ by

$$h(e) = e_\ell.$$

If $e = (x, y)$ is of type ℓ, we have

$$y_\ell = x_\ell + 1, \quad y_{\ell-1} = x_{\ell-1}, \quad y_{\ell+1} = x_{\ell+1} - 1. \tag{6.3.1}$$

We denote by $E_k^{(\ell)}$ the set of edges of type ℓ. Remark that the height of an edge does not characterize its position, since for example the translations of an edge of type 1 by $e^{i\pi/3}$ will have the same height.

Definition 6.3.3 (Discrete boundary)**.** The *boundary* ∂T_N of the triangular grid T_N is the set of edges (x, y) lying on the boundary of the triangle $[0, N] \cup [N, Ne^{i\pi/3}] \cup [0, Ne^{i\pi/3}]$.

The boundary ∂T_N can be decomposed into three subsets $\partial T_N^{(i)}$, $0 \leqslant i \leqslant 2$, where each set $\partial T_N^{(i)}$ consists of edges of type i. The coordinates of the corresponding edges are then the following.

$$\begin{aligned}
\partial T_N^{(0)} &= \left((r, N-r, 0), 0 \leqslant r \leqslant N-1\right), \\
\partial T_N^{(1)} &= \left((0, r, N-r), 0 \leqslant r \leqslant N-1\right), \\
\partial T_N^{(2)} &= \left((N-r, 0, r), 0 \leqslant r \leqslant N-1\right).
\end{aligned}$$

6.3.2 Puzzles and the quantum-LR coefficients

We will mainly work on puzzles describing the two-step flag cohomology from [Buc+16], in the special case where they describe the quantum Littlewood-Richardson coefficients previously introduced in Section 6.2.

Let us consider the set of puzzle pieces given in Figure 6.8, which are considered as the assignment of a label in $\{0, \dots, 7\}$ to edges of T_N around a triangular face. Each piece can be rotated by a multiple of $\frac{\pi}{3}$ but not reflected.

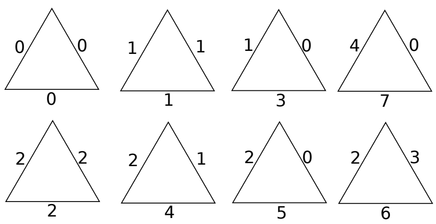

Figure 6.8: Possible pieces of the puzzle

Definition 6.3.4. A *triangular puzzle* of size $N \geqslant 1$ is a map $P : E_N \to \{0,\dots,7\}$ such that the value around each triangular face belongs to the set of possible puzzle piece displayed in Figure 6.8.
The *boundary coloring* ∂P of a puzzle P is the sequence $(\omega_0,\omega_1,\omega_2)$ such that ω_ℓ is the sequence $(P(e))_{e\in\partial^{(\ell)}T_N}$, where $\partial^{(\ell)}T_N$ is the sequence of boundary edges of T_N of type ℓ ordered by their height.

For any triple $(\omega_0,\omega_1,\omega_2)$ of words in $\{0,1,2\}^N$, we denote by $P(\omega_0,\omega_1,\omega_2)$ the set of puzzles whose boundary coloring is $(\omega_0,\omega_1,\omega_2)$. For $0 \leqslant k_0 \leqslant k_0+k_1 \leqslant N$, denote by $F(k_0,k_1,N)$ the two-step flag manifold

$$F(k_0,k_1,N) \coloneqq \{V_0 \subset V_1 \subset \mathbb{C}^N, \dim V_0 = k_0, \dim V_1 = k_0+k_1\}.$$

The cohomology ring $H^*F(k_0,k_1,N)$ admits a basis $\{\sigma_\omega\}$ of Schubert cycles indexed by words in $\{0,1,2\}^N$ with k_0 occurrences of 0 and k_1 occurrences of 1, see Section 3.2.4. Proving a conjecture of Knutson, it has been shown in [Buc+16] that the previously constructed puzzles describe the structure constants of $H^*F(k_0,k_1,N)$.

Theorem 6.3.5 ([Buc+16])**.** *For any triple $(\omega_0,\omega_1,\omega_2)$ of words in $\{0,1,2\}^N$ with same number of occurrences k_0 of 0 and k_1 of 1,*

$$\langle \sigma_{\omega_0}\sigma_{\omega_1}\sigma_{\omega_2}, \sigma_0\rangle_{H^*F(k_0,k_1,N)} = \#P(\omega_0,\omega_1,\omega_2),$$

where σ_0 is the fundamental class of $H^*F(k_0,k_1,N)$.

Thanks to a previous work [Buc03] relating the quantum cohomology of Grassmannians to the classical cohomology of the two-step flag manifold, Theorem 6.3.5 yields a similar expression in terms of puzzles for the quantum Littlewood-Richardson coefficients.

Corollary 6.3.6 ([Buc+16])**.** *Let $1 \leqslant n \leqslant N$ and $\lambda^0,\lambda^1,\lambda^2$ be partitions of length n with first part smaller than $N-n$ such that $|\lambda^1|+|\lambda^2| = |\lambda^0|+Nd$. Then, $c^{\lambda^0,d}_{\lambda^1,\lambda^2} = \#P(\omega_0,\omega_1,\omega_2)$, where $\omega_\ell, \ell\in\{0,1,2\}$ are constructed as follows :*

1. *for $\ell\in\{1,2\}$, set $\omega_\ell(\lambda^\ell_i+(n-i)) = 0$ for $1\leqslant i\leqslant n$ and $\omega_\ell(i) = 2$ otherwise,*
2. *set $\omega_0(N-1-(\lambda^0_i+(n-i))) = 0$ for $1\leqslant i\leqslant n$ and $\omega_0(i) = 2$ otherwise,*
3. *for $\ell\in\{0,1,2\}$, replace the d last occurrences of 0 and the d first occurrences of 2 in ω_ℓ by 1.*

The goal of this section and the next one is then to give a convex formulation of the latter results, yielding Theorem 6.4.3 for the expression of the structure constants of $H^*F(k_0,k_1,N)$ and Corollary 6.4.4 for the corresponding result concerning the quantum Littlewood-Richardson coefficients.

6.3.3 Edge, vertex and face partitions

The set of pieces can be further simplified in two steps by gluing some pieces along edges having same labels. First, gluing two pieces with edges having label 2 along a common edge labeled $4,5$ or 6 yields lozenges with edge labelled 2 and either $0,1$ or 3; then concatenating consecutively such lozenges on edges having same label $0,1$ or 3 and considering also the triangle with all edges labeled 2 yield the pieces of Figure 6.10, which are called pieces of type II. Let us then call pieces of type I any piece displayed in Figure 6.9 which consists of the first three triangles of Figure 6.8 and pieces obtained by concatenating two pieces

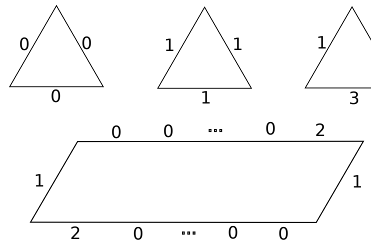

Figure 6.9: Puzzle pieces of type I.

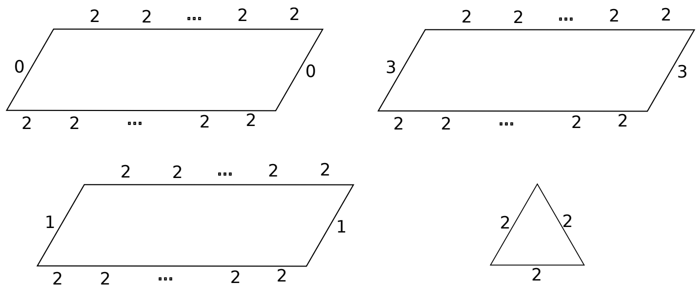

Figure 6.10: Puzzle pieces of type II.

with label $(2, 4, 1)$ and an arbitrary even number of pieces with label $(4, 7, 0)$. We will first show that a puzzle P is completely characterized by the position of pieces of type I.
Let us first mention a first general result on height of edges on the border of a same triangle.

Lemma 6.3.7 (Triangle sum)**.** *Suppose that f is a face of T_N with edges e^0, e^1, e^2. Then,*

$$\begin{aligned} h(e^0) + h(e^1) + h(e^2) &= N - 1 \text{ if } f \text{ is direct} \\ h(e^0) + h(e^1) + h(e^2) &= N - 2 \text{ if } f \text{ is reversed.} \end{aligned} \tag{6.3.2}$$

Moreover, $e^i_{i-1} = e^{i-1}_{i-1}$ (resp. $e^i_{i-1} = e^{i+1}_{i-1}$), for $i \in \{0, 1, 2\}$ if f is direct (resp. reversed).

Proof. A direct triangle has edges $e^0 = (N-(i+j+1), i+1, j)$, $e^1 = (N-(i+j+1), i, j+1)$ and $e^2 = (N - (i + j), i, j)$ for some $0 \leqslant i, j \leqslant N - 1$ with $i + j \leqslant N - 1$, so that

$$h(e^0) + h(e^1) + h(e^2) = N - 1 + 0 + 0 = N - (i + j + 1) + i + j = N - 1.$$

A reversed triangle has edges $e^0 = (N - (i + j + 1), i, j + 1)$, $e^1 = (N - (i + j), i - 1, j + 1)$ and $e^2 = (N - (i + j), i, j)$ for some $1 \leqslant i, j \leqslant N - 1$ with $i + j \leqslant N - 1$, so that

$$h(e^0) + h(e^1) + h(e^2) = N - 1 + 0 + 0 = N - (i + j + 1) + i - 1 + j = N - 2.$$

It is also clear from the coordinates of the edges that $e^i_{i-1} = e^{i-1}_{i-1}$ (resp. $e^i_{i-1} = e^{i+1}_{i-1}$) for $i \in \{0, 1, 2\}$ if the triangle is direct (resp. reversed). □

Definition 6.3.8 (Edge set, vertex, edge and face partitions)**.** The *edge set* of a puzzle P is the set $\mathcal{E}$ of edges labeled $0, 1, 3$ of either a type I piece or on the boundary.
The *vertex partition* of $\mathcal{E}$ is the covering $\mathcal{P}_v$ of $\mathcal{E}$ whose sets of size greater than one consist of all the edges colored $\{0, 1, 3\}$ of a same type I piece and singletons consists of edges of $\mathcal{E}$ on the boundary of T_N not belonging to a type I piece.
The *edge partition* of $\mathcal{E}$ is the set partition $\mathcal{P}_e$ of $\mathcal{E}$ whose blocks of size greater than one consist of edges of a common type II piece.
The *face partition* $\mathcal{P}_f$ is the set partition of V_N whose blocks are the connected components of the subgraph of T_N obtained by only keeping the edges colored 2.

Remark that $\mathcal{P}_e$ can also have singletons. An element $e \in \mathcal{E}$ is a singleton of $\mathcal{P}_e$ if and only if it is a common edge of two type I pieces. However, no element of $\mathcal{E}$ can be a singleton of both $\mathcal{P}_v$ and $\mathcal{P}_e$, since a border edge colored 0 or 1 not belonging to a type I piece has to belong to a type II piece.

Lemma 6.3.9 (Blocks of $\mathcal{P}_v$)**.** *A block of order* 3 *in* $\mathcal{P}_v$ *consists of three edges* e^0, e^1, e^2 *of type* $0, 1, 2$ *such that*

- *either* $e^i_{i-1} = e^{i-1}_{i-1}$, $i \in \{0,1,2\}$ *and* $\sum_{i=0}^2 h(e^i) = N-1$, *or* $e^i_{i-1} = e^{i+1}_{i-1}$, $i \in \{0,1,2\}$ *and* $\sum_{i=0}^2 h(e^i) = N-2$,
- $(c(e^0), c(e^1), c(e^2))$ *is either* $(0,0,0)$, $(1,1,1)$ *or any cyclic permutation of* $(0,1,3)$.

A block of order $2(r+1)$, $r \geqslant 2$ *in* $\mathcal{P}_v$ *consists of* $2(r+1)$ *edges* $\{e^0, f^1, \dots, f^r, e^{0'}, f^{1'}, \dots, f^{r'}\}$ *such that*

- $e^0, e^{0'}$ *are of type* i *and* $f^1, \dots, f^r, f^{1'}, \dots, f^{r'}$ *are of type* $i+1 \mod 3$ *for some* $i \in \{0,1,2\}$,
- $h(e^0) = h(e^{0'})$ *and* $f^1_i = \dots = f^r_i = f^{1'}_i + 1 = \dots = f^{r'}_i + 1 = h(e^0) + 1$,
- $h(f^s) = h(f^{s'}) = h(f^1) + (s-1) = e^{0'}_{i+1}$ *for* $1 \leqslant s \leqslant r$,
- *the edges* f^0 *and* $f^{r+1'}$ *of type* $i+1$ *with* $f^{r+1}_i = f^{r+1'}_i + 1 = h(e^0) + 1$ *and* $h(f^0) = e^{0'}_{i+1} - 1$ *and* $h(f^{r+1'}) = e^0_{i+1} - 1$ *are not in* $\mathcal{E}$.

Any edge $e \in \mathcal{E}$ *belongs to at most two blocks of* $\mathcal{P}_v$.

Proof. In the case of a block of order 3, $\{e^0, e^1, e^2\}$ is a triangle of T_N and the results on the height of the edges is given by Lemma 6.3.7. The results on the color of the edges is given by the possible coloring of edges of Type I pieces from Figure 6.9.
In the case of a block B of order $2(r+1)$, the edges correspond to the boundary edges not colored 2 of a puzzle piece of the last shape of Figure 6.9. We can thus first label cyclically the boundary edges colored 0 and 1 as $\{e^0, f^1, \dots, f^r, e^{0'}, f^{1'}, \dots, f^{r'}\}$ so that e^0 and $e^{0'}$ are colored 1 and of type $i \in \{0,1,2\}$ and $f^i, f^{i'}$, $1 \leqslant i \leqslant r$ are colored 0 and are of type $i+1$. Then, remark that such a piece is the concatenation of $r+1$ direct triangles $T_1, \dots, T_{r+1}$ and $r+1$ reversed triangles $T'_1, \dots, T'_{r+1}$ such that T_i and T'_i (resp. T'_i and T_{i+1}) share an edge of type $i+1$ (resp. $i-1$), e^0 (resp. $e^{0'}$ is the edge of type i of T_1 (resp. T'_{r+1}) end f^i (resp. $f^{i'}$) is the edge of type $i+1$ of T'_i (resp. T_{i+1}). The relations giving the height and the labels of the edges are then direct consequences of Lemma 6.3.7. ☐

Definition 6.3.10 (Admissible pair and strip)**.** A pair $B = \{e^1, e^2\} \subset E_N$ of edges is called *admissible* if e^1 and e^2 have same type $j \in \{0,1,2\}$ and same height.
The *strip* S_B of an admissible pair $B = \{e^1, e^2\}$ of type j is the set of all edges $e = (x,y) \in E_N$ such that x, y belong to the parallelogram delimited by e^1 and e^2. Namely, if $e^1_j = e^2_j$ and $e^2_{j-1} \leqslant e^1_{j-1}$, S_B is the set of edges (x,y) such that $e^1_j \leqslant x_j, y_j \leqslant e^1_j + 1$, $e^2_{j-1} \leqslant x_{j-1}, y_{j-1} \leqslant e^1_{j-1}$.
The *boundary* $\partial_2 S_B$ of a strip S_B consists of all edges of S_B type $j+1$.

Remark that such a definition is still valid if $e^1 = e^2$, in which case $B = \{e^1\}$ is always admissible and $S_B = \{e^1\}$. In particular, for any $B \in \mathcal{P}_e$ for a puzzle P, S_B consists of all edges appearing on the type II piece bordered by elements of B: B coincide exactly the boundary edges of S_B which are not labeled 2 and $\partial_2 S_B$ consists of the boundary edges of the type II piece which are labeled 2 (that is, all boundary edges except for B).

Moreover, given a pair $B = \{e^1, e^2\}$ of edges of same type j and same height, one can rephrase the condition of belonging to the strip S_B depending on the type of the edge we consider:

- if e has type j, $e \in S_B$ if and only if $e_j = e^1_j$ and $e_{j-1} \in [e^2_{j-1}, e^1_{j-1}]$,
- if e has type $j+1$, $e \in S_B$ if and only if $e_j \in \{e^1_j, e^1_j + 1\}$ and $e_{j-1} \in]e^2_{j-1}, e^1_{j-1}]$,
- if e has type $j-1$, $e \in S_B$ if and only if $e_j = e^1_j + 1$ and $e_{j-1} \in [e^2_{j-1}, e^1_{j-1}[$.

Lemma 6.3.11 (Edge partition pairs are admissible)**.** *Let P be a puzzle and $\mathcal{P}_e$ the corresponding edge partition. Any pair $\{e^1, e^2\} \in \mathcal{P}_e$ (with possibly $e^1 = e^2$) is admissible and is such that $C(e^1) = C(e^2) \in \{0, 1, 3\}$. For any different blocks $B, B' \in \mathcal{P}_e$, $S_B \cap S_{B'} \subset \partial_2 S_B \cap \partial_2 S_{B'}$.*

Proof. Recall that blocks of $\mathcal{P}_e$ of size greater that one corresponds to edges colored $0, 1$ or 3 on the border of a same Type II piece from 6.10. In particular the blocks of size greater than ones are only pairs, and the first part of the lemma is a direct consequence of the possible type II pieces of Figure 6.10.

For the second part of the lemma, remark first that for any block $\{e^1, e^2\}$ in $\mathcal{P}_e$, $S_B \cap \mathcal{E} = B$: first, boundary edges of S_B are either colored $0, 1$ or 3 and in B or colored 2 and not in $\mathcal{E}$. Then, interior edges of S_B which are colored $0, 1$ or 3 are boundary edges of two pieces with the same labelling of the second row of Figure 6.8 and thus are not in $\mathcal{E}$ (remark that the second piece of the second row of Figure 6.8 is used in the last type I piece of Figure 6.8 but is surrounded by pieces with different boundary labels).

Consider two different blocks $B, B' \in \mathcal{P}_e$. Let $e \in S_B \cap S_{B'}$. Remark that any edge of S_B or $S_{B'}$ is not on the border of the strip if and only if it is neither in $B \cup B'$ nor colored 2. Hence, if e is colored 2, then $e \in \partial_2 S_B \cap \partial_2 S_{B'}$. Suppose by contradiction that e is not colored 2. Since $B \cap S_{B'} = S_B \cap B' = \emptyset$ and e is not colored 2, e belongs to the interior of both S_B and $S_{B'}$: hence, the two triangular puzzle pieces whose boundary is e belong to S_B and $S_{B'}$, and we deduce that there is an edge f^1 colored $0, 1$ or 3 which belong to $S_B \cap S_{B'}$. If $f^1 \notin B \cup B'$, f^1 belong to the interior of S_B and $S_{B'}$, and thus there exists an edge $f^2 \in S_B \cap S_{B'}$ with $f^2_{i+1} = f^1_{i+1} + 1$. Let us repeat the process until there is an edge $f^s \in S_B \cap S_{B'}$ which belongs to either B or B'. This means that $B \cap S_{B'}$ or $B' \cap S_B$ is not empty, which contradicts the fact proven previously that $S_B \cap \mathcal{E} = B$ and $S_{B'} \cap \mathcal{E} = B'$. Hence, any edge not colored 2 does not belong to $S_B \cap S_{B'}$. ☐

Definition 6.3.12 (Crossing pairs)**.** We say that two admissible pairs $B = \{e^1, e^2\}$ and $B' = \{e^3, e^4\}$ of E_N *cross* when

- either $B \cap S_{B'} \neq \emptyset$ or $B' \cap S_B \neq \emptyset$,
- or $B = \{e^1, e^2\}, B' = \{e^3, e^4\}$ are blocks of size two which are of respective type $j, j+1$ for some $j \in \{0, 1, 2\}$, and

$$e^3_j \leqslant e^1_j = e^2_j \leqslant e^4_j \quad e^1_{j+1} \leqslant e^3_{j+1} = e^4_{j+1} < e^2_{j+1}.$$

Lemma 6.3.13 (Non-crossing condition)**.** *For any distinct pairs $B, B' \subset E_N$, the two following properties are equivalent :*

1. *B and B' do not cross,*

2. $S_B \cap S_{B'} \subset \partial_2 S_B \cap \partial_2 S_{B'}$.

Proof. Let B, B' be admissible pairs of E_N.
If B is a singleton, then $\partial_2 S_B = \emptyset$ and thus $\partial_2 S_B \cap \partial_2 S_{B'} = \emptyset$. Moreover, since B is a singleton, $B = S_B$. Hence, the non-crossing condition is equivalent to $S_B \cap S_{B'} = \emptyset$, and B and B' do not cross if and only if $S_B \cap S_{B'} \subset \emptyset = \partial_2 S_B \cap \partial_2 S_{B'}$.
If $B = \{e^1, e^2\}$ and $B' = \{e^3, e^4\}$ are pairs of the same type $j \in \{0, 1, 2\}$, the non-crossing condition means that $B \cap S_{B'} = \emptyset$ and $B' \cap S_B = \emptyset$. Suppose that B and B' cross, and without loss of generality, assume that $B \cap S_{B'} \neq \emptyset$. Since $B \cap \partial_2 S_B = \emptyset$, we deduce that $B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$. Hence, $S_B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$.

Reciprocally, suppose that $S_B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$, and assume without loss of generality that $S_B \cap S_{B'} \cap (S_B \setminus \partial_2 S_B) \neq \emptyset$. Since $\partial_2 S_B$ is the set of edges of S_B of type $j+1$, there exists an edge $e = (x, y)$ of type j or $j-1$ in $S_B \cap S_{B'}$. If e is of type j, this means that $e_j = e^1_j = e^2_j$ and $e_{j+1} \in [e^1_{j+1}, e^2_{j+1}]$. Similarly, $e_j = e^3_j = e^4_j$ and $e_{j+1} \in [e^3_{j+1}, e^4_{j+1}]$. Hence, $[e^1_{j+1}, e^2_{j+1}] \cap [e^3_{j+1}, e^4_{j+1}] \neq \emptyset$, and the extremity of one of these intervals is contained in the other. Assume without loss of generality that $e^1_{j+1} \subset [e^3_{j+1}, e^4_{j+1}]$. Then, e_1 is an edge of type j such that $e^1_j = e^3_j = e^4_j$ and $e^1_{j+1} \in [e^3_{j+1}, e^4_{j+1}]$, thus $e^1 \in S_{B'}$ and thus $B \cap S_{B'} \neq \emptyset$. If $e = (x, y)$ is of type $j-1$, then $x_{j-1} = y_{j-1} - 1$ and $x_j = y_j + 1$, see (6.3.1). Hence, the conditions $x_j, y_j \in \{e^1_j, e^1_j + 1\}$ and $x_{j-1}, y_{j-1} \in [e^2_{j-1}, e^1_{j-1}]$ from Definition 6.3.10 yield that $e' = (x', x)$ with $x'_j = y_j$ and $x'_{j-1} = x_j$ is an edge of type j which belongs to S_B. Similarly, $e' \in S_{B'}$, and the previous reasoning allows to conclude that $B \cap S_{B'} \neq \emptyset$ or $B' \cap S_B \neq \emptyset$.
Suppose finally that $B = \{e^1, e^2\}$ and $B' = \{e^3, e^4\}$ are pairs of respective type j and $j+1$ for $j \in \{0, 1, 2\}$. Remark first that edges of $\partial_2 S_B$ have type $j+1$ and edges of $\partial_2 S_{B'}$ have type $j+2$, so that $\partial_2 S_B \cap \partial_2 S_{B'} = \emptyset$.
Suppose that B and B' cross. First, if $B \cap S_{B'} \neq \emptyset$ or $B' \cap S_{B'} \neq \emptyset$, then $S_B \cap S_{B'} \neq \emptyset$ and thus $S_B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$. Suppose that $B \cap S_{B'} = B' \cap S_{B'} = \emptyset$, and thus

$$e^3_j \leqslant e^1_j = e^2_j \leqslant e^4_j \quad e^1_{j+1} \leqslant e^3_{j+1} = e^4_{j+1} < e^2_{j+1}.$$

Consider the edge $e = (x, y)$ of type $j+1$ with $x_j = e^1_j$ and $x_{j+1} = e^3_{j+1}$. Since e is of type $j+1$, $y_j = x_j$ and $y_{j+1} = x_{j+1} + 1$. First, since $x_{j+1}, y_{j+1} \in \{e^3_{j+1}, e^3_{j+1} + 1\}$ and $x_j = y_j = e^1_j \in [e^3_j, e^4_j]$, $e \in S_{B'}$. Then, remark that

$$x_{j-1} = N - (x_j + x_{j+1}) = N - e^1_j - e^3_{j+1},\ e^1_{j-1} = N - e^1_j - e^1_{j+1},\ e^2_{j-1} = N - e^2_j - e^2_{j+1}.$$

From the equality $e^1_j = e^2_j$ and $e^1_{j+1} \leqslant e^3_{j+1} < e^2_{j+1}$, we deduce that $x_{j-1} \in]e^2_{j-1}, e^1_{j-1}]$. Since $y_{j-1} = N - y_j - y_{j+1} = N - x_j - x_{j+1} - 1 = x_{j-1} - 1$, $y_{j-1} \in [e^2_{j-1}, e^1_{j-1}]$. The two latter inclusions together with $y_j = x_j = e^1_j$ yield that $e \in S_B$. In particular, $S_B \cap S_{B'} \neq \emptyset$ and thus $S_B \cap S_{B'} \not\subset \partial_2 S_B \cap \partial_2 S_{B'}$.

Suppose that $S_B \cap S_{B'} \neq \emptyset$. If $B \cap S_{B'} \neq \emptyset$ or $B' \cap S_B \neq \emptyset$, then B and B' cross. When $B \cap S_{B'} = B' \cap S_B = \emptyset$, then $S_B \cap S_{B'} \neq \emptyset$ if and only if $\partial_2 S_B \cap S_{B'} \neq \emptyset$. One implication is straightforward. For the other implication, remark that there is necessarily an edge $e \in S_B \cap S_{B'}$ which belongs to B or $\partial_2 S_B$, and since $B \cap S_{B'} = \emptyset$, $e \in \partial_2 S_B$. In particular, since B has type j, edges of $\partial_2 S_B$ have type $j+1$ and thus e has type $j+1$. The edge $e = (x, y)$ of type $j+1$ belongs to $\partial_2 S_B$ if and only

$$\begin{cases} x_j = y_j \in \{e^1_j, e^1_j + 1\}, \\ x_{j-1} \in [e^2_{j-1}, e^1_{j-1}] \text{ and } y_{j-1} = x_{j-1} - 1 \in [e^2_{j-1}, e^1_{j-1}]. \end{cases}$$

Similarly e of type $j+1$ to belong to $S_{B'}$ if and only if

$$\begin{cases} x_{j+1} = y_{j+1} - 1 = e^3_{j+1} = e^4_{j+1}, \\ x_j = y_j \in [e^3_j, e^4_j]. \end{cases}$$

Hence, $e \in \partial_2 S_B \cap S_{B'}$ if and only if

$$\begin{cases} x_j = y_j \in \{e^1_j, e^1_j + 1\} \cap [e^3_j, e^4_j],\ x_{j+1} = y_{j+1} - 1 = e^3_{j+1} \\ N - x_j - e^3_{j+1} \in]N - e^2_j - e^2_{j+1}, N - e^1_j - e^1_{j+1}]. \end{cases}$$

If $x_j = e^1_j$, the latter conditions imply

$$\begin{cases} e^3_j \leqslant e^1_j = e^2_j \leqslant e^4_j, \\ e^1_{j+1} \leqslant e^3_{j+1} = e^4_{j+1} < e^2_{j+1}. \end{cases}$$

If $x_j = e^1_j + 1$, the latter conditions yield

$$\begin{cases} e^3_j - 1 \leqslant e^1_j = e^2_j \leqslant e^4_j - 1,\ x_{j+1} = y_{j+1} - 1 = e^3_{j+1} \\ e^1_{j+1} - 1 \leqslant e^3_{j+1} = e^4_{j+1} < e^2_{j+1} - 1. \end{cases}$$

If $e^3_j - 1 = e^2_j$ and $e^1_{j+1} - 1 \leqslant e^3_{j+1} < e^2_{j+1} - 1$, then $e^3_j = e^2_j + 1$ and

$$e^3_{j-1} = N - e^3_j - e^3_{j+1} \in]N - e^2_j - e^2_{j+1}, N - e^2_j - e^1_{j+1}] =]e^2_{j-1}, e^1_{j-1}],$$

so that $e^3 \in S_B$ by the condition following Definition 6.3.10. Likewise, if $e^3_{j+1} = e^1_{j+1} - 1$ and $e^1_j \in [e^3_j, e^4_j]$, then $e^1 \in S_{B'}$. Hence, the fact that $B \cap S_{B'} = B' \cap S_B = \emptyset$ strengthens the above condition to imply

$$\begin{cases} e^3_j \leqslant e^1_j = e^2_j \leqslant e^4_j, \\ e^1_{j+1} \leqslant e^3_{j+1} = e^4_{j+1} < e^2_{j+1}. \end{cases}$$

□

From the latter lemma, we deduce a description of puzzles in terms of their type I pieces.

Proposition 6.3.14 (Partitions determine puzzles)**.** *The map $\Phi : P \mapsto (\mathcal{E}, c)$ is a bijection from the set of puzzles to the set of subsets of E_N with a coloring $c : \mathcal{E} \to \{0, 1, 3\}$ such that there exist a covering $\mathcal{P}_v$ and a partition $\mathcal{P}_e$ of $\mathcal{E}$ with*

- *blocks of $\mathcal{P}_v$ are of size 3 or $2(r+1)$, $r \geqslant 1$ and satisfy the properties of Lemma 6.3.9,*
- *blocks of $\mathcal{P}_e$ are either singletons or pairs satisfying the properties of Lemma 6.3.11,*
- *If e belong to only one block of $\mathcal{P}_v$, then e is not a singleton of $\mathcal{P}_e$ and if $e \neq e'$ belong to a same block of $\mathcal{P}_v$, then $\{e, e'\} \notin \mathcal{P}_e$.*

Proof. Let us build the candidate inverse bijection, and consider a subset $\mathcal{E} \subset E_N$ with a coloring $c : \mathcal{E} \to \{0, 1, 3\}$ and a covering $\mathcal{P}_v$ and a partition $\mathcal{P}_e$ satisfying the conditions of Proposition 6.3.14. For each pair $B = (e, e') \in \mathcal{P}_e$ of type i and color c, color all edges of type i (resp. $i+1$, resp. $i+2$) of the strip $S_B \setminus \{e, e'\}$ with the color c (resp. 2, resp. $c+5$). Remark that such a coloring is possible, since by properties of Lemma 6.3.11, any edge belonging to $S_B \cap S_{B'}$ for two strips of respective types i, i' must be included in the boundary $\partial_2 S_B \cap \partial_2 S_{B'}$, which consists then of edges of same type $i+1 = i'+1$ colored 2 by the above rule.

Then, consider any block $B \in \mathcal{P}_v$ of order $2(r+1)$ whose edges e, e' colored 1 are of type i. Remark then that by the properties of Lemma 6.3.9, $B = \{e, e'\}$ is an admissible pair of edges of $\mathcal{E}$ of same color 1. Moreover, all edges but two of the boundary $\partial_2 S_B$ of the strip S_B consists of edges of $\mathcal{E}$ colored 0.

Let i be the type of B, and suppose by contradiction that there is an edge e^0 of type i or $i-1$ inside the strip S_B which is contained in a strip $S_{\tilde{B}}$ for some $\tilde{B} \in \mathcal{P}_e$. Suppose first that $\tilde{B} \neq \{e^0\}$. Then, since e^0 shares a vertex with at least three edges of type $i+1$ colored 0, $S_{\tilde{B}}$ must contain one of those three edges, called f^1; since $\mathcal{P}_e$ is non-crossing and $f^1 \in \mathcal{E} \cap S_{\tilde{B}}$, $f^1 \in \tilde{B}$ and thus $\tilde{B}$ is of type $i+1$. Since $e^0 \in S_{\tilde{B}}$ and $\tilde{B}$ is of type $i+1$, the other edge f^2 of S_B with same height as f^1 must also belong to $S_{\tilde{B}}$, and by the non-crossing condition we have $\tilde{B} = \{f^1, f^2\}$. This contradicts the fact that two elements of the same block of $\mathcal{P}_v$ do not belong to the same block of $\mathcal{P}_e$.

If e^0 is a singleton of $\mathcal{P}_e$, then $e \in \mathcal{E}$ and belongs to at least two blocks of $\mathcal{P}_v$. Hence, there must be another edge e^1 between e^0 and e in the strip S_B which belongs to $\mathcal{E}$. Iterating the process yields an edge $\tilde{e}$ such that e and $\tilde{e}$ belong to a same block $\tilde{B}$ in $\mathcal{P}_v$. Then, $\tilde{e}$ cannot be of type $i-1$ otherwise this block would be a block of order 3, contradicting the fact that the edge f of type $i+1$ with $f_i = e_i, f_{i+1} = e_{i+1} - 1$, which would then belong to this block $\tilde{B}$, does not belong to $\mathcal{E}$. Similarly, if $\tilde{e}$ is of type i, then e and $\tilde{e}$ would be boundary edges of type I piece with $2(r'+1)$ edges, with $1 \leqslant r' < r$. This is impossible since the boundary $\partial_2 S_{\{e,\tilde{e}\}}$ contains at most one edge which is not in $\mathcal{E}$. Hence, no edge of type i or $i-1$ inside S_B belongs to some strip S_B for $B \in \mathcal{P}_e$ and thus none of those edges has been colored 2 in the previous labelling. Therefore, one can color all type i edge in $S_{\tilde{B}}$ different from e, e' with the label 7 and all type $i-1$ edge in $S_{\tilde{B}}$ with label 6.

Finally, the edge f of type $i+1$ with $f_i = e_i, f_{i+1} = e_{i+1} - 1$ can not be part of a strip $S_{\tilde{B}}$ for some $\tilde{B} = (f^1, f^2)$ of type $i+1$ or $i-1$, for otherwise e would also belong to $S_{\tilde{B}}$ and $\mathcal{P}_e$ would not be non-crossing. Hence, either $f \in \partial_2 S_B$ for some strip S_B and f has been labelled 2 in the first coloring step, or f has not been colored before and thus f can be colored 2.

Color all remaining edges with the label 2. One then checks that the labels on the boundary of any triangle of the puzzle satisfy the conditions of Figure 6.8, so that the labelling of edges of E_N yields a genuine puzzle P. It is then straightforward to check that $P_v = \mathcal{P}_v$ and $P_e = \mathcal{P}_e$. The map Φ is thus surjective.

For the injectivity, remark that the data of $\mathcal{P}_v$ alone gives the list and position of type I pieces of the puzzle, which uniquely characterizes it. □

6.3.4 Graph of a puzzle

Definition 6.3.15 (Graph of a puzzle)**.** The *graph* of a puzzle P is the graph $\mathcal{G}_P$ whose set of vertices is P_v, set of edges is P_e and set faces is P_f.
The *endpoints* of an edge $B_e \in P_e$ are the vertices $B_v, B_{v'} \in P_v$ such that $B_e \cap B_v \neq \emptyset$ and $B_e \cap B_{v'} \neq \emptyset$.
The *boundary* of a face $B_f \in P_v$ are the edges $B \in P_e$ such that there is $e \in B, v \in B_f$ such that v is an endpoint of e. A face $B_f \in P_f$ is called an outer face (resp. inner face) if there is an element (resp. no element) $v \in B_f$ on the border of T_N.

Remark that elements of P_v and P_e are sets of edges of T_N while elements of P_f are set of vertices of T_N. Moreover, any edge $B \in P_e$ has a type $\ell \in \{0, 1, 2\}$ and a color $c \in \{0, 1, 3\}$,

which is the type and the color of the edges of T_N in B.
Let $B_f \in \mathcal{P}_f$ and denote by ∂B_f the set of edges on the boundary of B_f. Then, there is a natural cyclic order on ∂B_f such that $\partial B_f = (B_1 < \ldots < B_p)$ where B_i and B_{i+1} share a vertex of $\mathcal{P}_v$ and the edges of ∂B_f are read in the clockwise order around the region B_f.

Lemma 6.3.16 (Type of face boundaries)**.** *Let $B_f \in \mathcal{P}_f$. Then, the sequence of type of edges on the boundary of B_f is a subsequence of $(0,1,2,0,1,2)$ (up to cyclic rotation), and two consecutive edges $B < B'$ on the boundary of B_f sharing a vertex $B_v \in \mathcal{P}_v$ are*

- *of type $(\ell, \ell+1)$ if B_v is a block of size three and the color of B and B' are either both 0, both 1 or $(3,0)$, $(0,1)$ or $(1,3)$, or a block of size $2r$, $r \geqslant 2$ and B, B' have respective color $(0,1)$.*
- *of type $(\ell, \ell+2)$ if B_v is a block of size $2r$, $r \geqslant 2$ and B and B' have color $(0,1)$,*
- *of type ℓ if B_v is a block of size $2r$, $r \geqslant 3$ and B and B' have color 0.*

Proof. Let $(B_1, \ldots, B_p)$ be the previously defined cyclic ordering of the edges around B_f such that B_i, B_{i+1} share a vertex in $\mathcal{P}_v$. Let $1 \leqslant i \leqslant p$ and denote by ℓ the type of B_i. Since B_i, B_{i+1} share the vertex B_v, there exist $e^i \in B_i$ and $e^{i+1} \in B_{i+1}$ such that $e^i, e^{i+1} \in B_v$. Since e^i, e^{i+1} are not colored 2, the type and colors of e^i (resp. e^{i+1}) are the ones of B_i (resp. B_{i+1}).
If B_v is a block of size 3, then it is a triangle vertex whose boundary colors in the clockwise order are either $(0,0,0)$, always $(1,1,1)$, or $(1,3,0)$ up to a rotation. Since the angle between e^i and e^{i+1} is $-\pi/3$, the type of e_{i+1} is $\ell+1$, and the colors B_i, B_{i+1} are either $(0,0)$, $(1,1)$, $(3,0)$, $(0,1)$ or $(1,3)$.
If B_v is a block of size $2(r+1)$, $r \geqslant 1$, then 3 configurations can occur depending on the colors of the consecutive edges :

- if e^i is colored 1 and e^{i+1} is colored 0, then the type of e^{i+1} is $\ell+1$,
- if e^i is colored 0 and e^{i+1} is colored 1, then the type of e^{i+1} is $\ell-1$,
- if e^i and e^{i+1} are both colored 0 then the edges are adjacent and have same type ℓ. Remark that in this case, the vertex B_v must have at least 6 edges.

Finally, remark that the angle between two consecutive edges B_i, B_{i+1} is equal to $(1 - r_i/3)\pi$ if the difference of the type from B_i to B_{i+1} is r_i (with $r_i = 3$ if B_i and B_{i+1} have both type ℓ). Since the sum of the angles must be equal to the $(p-2)\pi$ if B_f is an inner face and smaller otherwise, we must have $\sum_{i=1}^{p}(1 - r_i/3) \leqslant p-2$, so that

$$\sum_{i=1}^{p} r_i \leqslant 6.$$

We deduce that the sequence of types of edges of the boundary must be a subsequence of $(0,1,2,0,1,2)$, up to cyclic permutation. $\square$

6.4 Discrete two-colored dual hive model

In this section, we associate to each puzzle of size N a two-colored hive in the same spirit as in [KT99] using the graph representation of puzzles from Section 6.3.4. Beware that because of the rigid crossings from Figure 6.9, the discrete hives will not be actual hives as in [KT99] but rather a dual hive. Let us fix in this section the number k of edges colored

0 or 1 on one edge of the puzzles. This number is the same on each edge of the puzzle and is part of the boundary data of the puzzle. The two colored dual hive associated to a puzzle will then be a decoration of the triangular grid T_k instead of T_N. All the notation introduced for T_N are thus still valid for T_k.

Definition 6.4.1 (Two-colored discrete dual hive)**.** A *two-colored discrete dual hive* of size (k, N) is given by the following combinatorial data on T_k:

- a *color map* $C : E_k \to \{0, 1, 3, m\}$, such that the boundary colors around each triangular face in the clockwise order is either $(0, 0, 0), (1, 1, 1), (1, 0, 3)$ or $(0, 1, m)$ up to a cyclic rotation.
- a *label map* $L : E_k \to \mathbb{N}$, with the two following conditions:
  1. for all $f \in F_k$ with boundary edges e^0, e^1, e^2, $L(e^0) + L(e^1) + L(e^2) = N - 1$ except when $f \in F_k^-$ with boundary colors different from $\{0, 1, m\}$, in which case $L(e^0) + L(e^1) + L(e^2) = N - 2$,
  2. if e, e' are edges of same type $\ell \in \{0, 1, 2\}$ on the boundary of a same lozenge, then
     (a) $L(e) = L(e')$ if the middle edge is colored m,
     (b) $L(e) \geqslant L(e')$ if $e'_{\ell+1} = e_{\ell+1} + 1$ and no edge different from e' is colored m,
     (c) $L(e) > L(e')$ if either $e_\ell > e'_\ell$ or if both $e'_{\ell+1} = e_{\ell+1} + 1$ and one of the boundary edges different from e' is colored m.

The *boundary value* $[(c^{(0)}, c^{(1)}, c^{(2)}), (l^{(0)}, l^{(1)}, l^{(2)})]$ of a two-colored discrete dual hive is the restriction of (C, L) to ∂T_k, where $c^{(i)} \in \{0, 1, 3, m\}^k$ (resp. $l^{(i)} \in \mathbb{N}^k$) is the restriction of C (resp. L) to $\partial T_k^{(i)}$, for $0 \leqslant i \leqslant 2$.

For $(c, l) = (c^{(0)}, c^{(1)}, c^{(2)}, l^{(0)}, l^{(1)}, l^{(2)}) \in \{0, 1\}^{3k} \times \mathbb{N}^{3k}$, we denote by $\mathcal{H}(c, l, N)$ the set of two-colored discrete dual hives with boundary value (c, l).

Remark 6.4.2. As a corollary of the Condition (2) on the label map, we have $L(e) > L(e')$ for any pair of edges e, e' of same type ℓ such that $e_\ell > e'_\ell$ and $e_{\ell+1} \leqslant e'_{\ell+1}$. Indeed, it suffices to show this for e, e' such that $e_\ell = e'_\ell + 1$ and $e'_{\ell+1} \in \{e_{\ell+1}, e_{\ell+1} + 1\}$. The case $e'_{\ell+1} = e_{\ell+1}$ is given by Condition (2.c), and we now suppose that $e'_{\ell+1} = e_{\ell+1} + 1$. Let e'' be such that $e''_{\ell+1} = e'_{\ell+1}$ and $e''_\ell = e_\ell = e'_\ell + 1$. If the middle edge of the lozenge with boundary e', e'' is not colored m, then by Condition (2.c) we have $L(e'') > L(e')$, and then by (2.a) or (2.b), we get $L(e) \geqslant L(e'') > L(e')$. If the middle edge of the lozenge with boundary e', e'' is colored m, then $L(e') = L(e'')$. Then, the middle edge colored m of this lozenge is then a boundary edge of the lozenge with boundary e'', e different from e'' and e, so that (2.c) implies that $L(e) > L(e'') = L(e')$.

For any triple $\omega = (\omega_0, \omega_1, \omega_2)$ of words $\{0, 1, 2\}^N$ with k_0 occurrences of 0 and k_1 occurrences of 1, denote by $(c(\omega), l(\omega))$ the sequence $(c^{(0)}, c^{(1)}, c^{(2)}, l^{(0)}, l^{(1)}, l^{(2)}) \in \{0, 1\}^{3k} \times \mathbb{N}^{3k}$ where $k = k_0 + k_1$ and $c^{(i)}$ is the word obtained from ω_i by deleting the letters 2 and $l^{(i)}$ is the sequence of positions of the letters 0 or 1 in ω_i. The following result gives a formulation of Theorem 3.2.12 in terms of integer points counting of polytopes.

Theorem 6.4.3 (Dual hive in the two-step case)**.** *For any triple $\omega = (\omega_0, \omega_1, \omega_2)$ of words $\{0, 1, 2\}^N$ with k_0 occurrences of 0 and k_1 occurrences of 1,*

$$\langle \sigma_{\omega_0} \sigma_{\omega_1} \sigma_{\omega_2}, \sigma_0 \rangle_{H^* F(k_0, k_1, N)} = \#\mathcal{H}(c(\omega), l(\omega), N). \tag{6.4.1}$$

Theorem 6.4.3 directly yields a similar expression of the quantum Littlewood–Richardson coefficients as the number of integer points in discrete dual hives. For three partitions λ, μ, ν of length n with first part smaller than $N-n$ and such that $|\lambda| + |\mu| = |\nu| + dN$, set

$$H(\lambda, \mu, \nu, N) = \mathcal{H}(c(\omega), l(\omega), N),$$

where ω is the triple of words in $\{0,1,2\}^N$ associated to $\lambda^1 = \lambda$, $\lambda^2 = \mu$, $\lambda^0 = \nu$ and for the corresponding d.

Corollary 6.4.4 (Dual hive for the q-LR coefficients)**.** *For λ, μ, ν of length n with first part smaller than $N-n$ and such that $|\lambda| + |\mu| = |\nu| + dN$,*

$$c_{\lambda,\mu}^{\nu,d} = \#H(\lambda, \mu, \nu, N).$$

The rest of this section is then devoted to a proof of Theorem 6.4.3, which is obtained by exhibiting a bijection $\zeta : P(\omega) \to \mathcal{H}(c(\omega), l(\omega), N)$.

Given a puzzle $P \in P(\omega)$, let $\mathcal{G}_P$ be the corresponding graph introduced in Section 6.3.4. Let us first transform the graph $\mathcal{G}_P$ into a new graph $\widehat{G}_P$ by blowing up each vertex $v \in \mathcal{P}_v$ of size $2(r+1)$ as follows.

Definition 6.4.5 (Blowup of vertex)**.** Let vertex $v \in \mathcal{P}_v$ be a vertex of size $2(r+1)$ with adjacent edges $(B^1, \ldots, B^{2r+2})$ (indexed in the cyclic order) such that B^1, B^{r+2} have type $\ell \in \{0,1,2\}$ and are colored 1 and $B^i, i \notin \{1, r+2\}$ have type $\ell+1$ and are colored 0. Introduce $2r-1$ new edges $\tilde{B}^1, \ldots, \tilde{B}^{2r-1}$ of type $\ell-1, \ell, \ldots, \ell-1$ and colored $m, 1, \ldots, m$ and transform v into $2r$ vertices $v^1, \ldots, v^{2r}$ such that the edges adjacent to v^{2j+1} are $(\tilde{B}^{2j}, B^{2r+2-j}, \tilde{B}^{2j+1})$ and edges adjacent to v^{2j+2} are $(\tilde{B}^{2j+1}, \tilde{B}^{2j+2}, B^{j+2})$ with the convention $\tilde{B}^0 = B^1$ and $\tilde{B}^{2r} = B^{r+2}$. We define the height of $\tilde{B}^i$ as $h(\tilde{B}^{2i}) = h(B^1)$ and $h(\tilde{B}^{2i-1}) = N - 1 - h(B^1) - h(B^{2i})$ for $1 \leqslant i \leqslant r-1$.

The resulting graph is called the *blowup* of v.

The picture of the blowup of a vertex of size 6 is given in Figure 6.11.

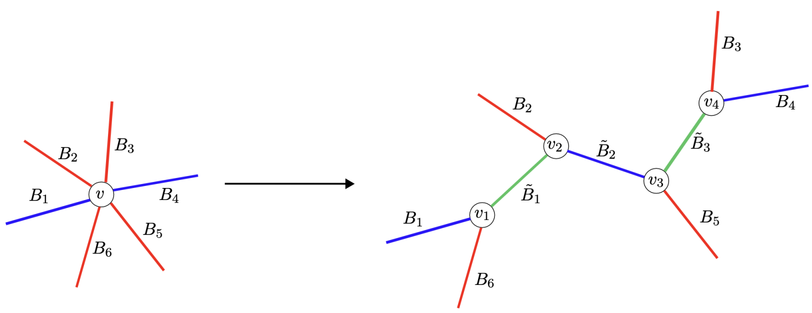

Figure 6.11: Blowup of a vertex of size 6 : edges in blue (resp. red, resp. green) are of type ℓ (resp. $\ell+1$, resp. $\ell+2$) and colored 1 (resp. 0, resp. m).

Definition 6.4.6 (Blowup of the graph of a puzzle)**.** The *blowup* $\widehat{G}_P$ of the graph $\mathcal{G}_P$ is the graph obtained by blowing up every vertex of size $2(r+1)$, $r \geqslant 1$.

Remark that the blowup of graph is well-defined because any vertex of size $2(r+1)$ of $\mathcal{G}_P$ has the form of Definition 6.4.5 thanks to Lemma 6.3.9. The blowup graph has then only vertices of degree 3 or singletons (which correspond to edges of T_N on the boundary of the triangle).

Lemma 6.4.7 (Faces in $\widehat{G}_P$)**.** *Let B_f be a face in $\widehat{G}_P$. Then, the boundary of B_f has*

- 6 *edges if no edge of the boundary of B_f is a boundary edge of $\mathcal{G}_P$,*
- 4 *edges if two edges of the boundary of B_f are boundary edges of $\mathcal{G}_P$ of the same type,*
- 2 *edges if two edges of the boundary of B_f are boundary edges of $\mathcal{G}_P$ of different type.*

Proof. By Lemma 6.3.16 and the blowing-up of vertices of degree larger than 4, two edges B, B' on the boundary of B_f sharing a vertex are of type $(\ell, \ell+1)$, and the edge type of boundary edges of B_f is a subsequence of $(0,1,2,0,1,2)$ (up to a cyclic rotation). The only possibility for B_f to have less than 6 edges on the boundary is then having edges which have a singleton as boundary vertex. The edge of $\mathcal{E}$ corresponding to this singleton is necessary a boundary edge of the graph and thus B_f is a connected component of $\mathcal{P}_2$ touching the boundary of T_N. There are then two possibilities : either B_f contain one of the three extreme vertex of T_N, in which case the boundary edges of B_f have different type and by convexity, B_f has only two edges in $\widehat{G}_P$, or B_f contains only boundary vertices which are not extreme points of T_N, in which case the boundary edges have same type ℓ and the boundary of B_f consists of four edges of type $(\ell, \ell+1, \ell+2, \ell)$. $\square$

Construction of a discrete two-colored dual hive from a puzzle

The resulting planar graph $\widehat{\mathcal{G}}_P$ is thus a graph with only trivalent vertices and hexagonal inner faces. From each side of the triangle T_N, there are $k-2$ faces $B \in \mathcal{P}_f$ which have degree 4 (one for each pair of consecutive boundary edge labeled 0 or 1 on a same side of T_N) and from each extreme vertex of T_N there is a face of degree 2.
Let us denote by $\widetilde{\mathcal{G}}_P$ the dual graph, namely the graph whose vertices are faces of $\widehat{\mathcal{G}}_P$, faces are vertices of $\widehat{\mathcal{G}}_P$ and such that there is one edge between each neighboring faces of $\widetilde{\mathcal{G}}_P$ (which correspond then to vertices of $\widehat{\mathcal{G}}_P$).

Lemma 6.4.8 (Dual graph to T_k)**.** *There is an isomorphism from $\widetilde{\mathcal{G}}_P$ to T_k mapping edges of type ℓ of $\widetilde{\mathcal{G}}_P$ to edges of type ℓ of T_k.*

Proof. Since vertices of $\widehat{\mathcal{G}}_P$ are trivalent, faces of $\widetilde{\mathcal{G}}_P$ are triangular. Similarly, inner faces of $\widehat{\mathcal{G}}_P$ have degree 6, and thus inner vertices of $\widetilde{\mathcal{G}}_P$ have degree 6. Hence, $\widetilde{\mathcal{G}}_P$ is isomorphic to a polygon H of the planar triangular grid. Since the sequence of degrees of the $3k$ outer faces of $\widehat{\mathcal{G}}_P$ is

$$(2, \underbrace{4, \ldots, 4}_{k \text{ times}}, 2, \underbrace{4, \ldots, 4}_{k \text{ times}}, 2, \underbrace{4, \ldots, 4}_{k \text{ times}}),$$

the same holds for the sequence of degrees of outer vertices of $\widetilde{\mathcal{G}}_P$. Remark that for each vertex of degree 4 (resp. 2) in $\widetilde{\mathcal{G}}_P$, the angle of the boundary at the corresponding vertex in H is π (resp. $5\pi/3$). We deduce that the boundary of the $\widetilde{\mathcal{G}}_P$ is isomorphic to the one of T_k, and thus $\widetilde{\mathcal{G}}_P$ is isomorphic to T_k. Let us denote by $\zeta : E(\widetilde{\mathcal{G}}_P) \to E_k$ the corresponding bijection between set of edges.
Remark that around every triangle of $\widetilde{\mathcal{G}}_P$ the type of the edges is $(\ell, \ell+1, \ell+2)$ (this is true for faces coming from trivalent vertex of $\mathcal{G}_P$ and true by construction for faces coming from the blowing up of higher order vertices of $\widehat{\mathcal{G}}_P$). We deduce that all edges with the same type in $\widetilde{\mathcal{G}}_P$ are sent through ζ to edges with the same orientation in T_k.

Up to composing ζ with an internal rotational symmetry of T_k, we can thus assume that ζ preserves the type of the edges. $\square$

Each edge B of $\widetilde{\mathcal{G}}_P$ has then a color $c(B)$ and a height $h(B)$ coming from the dual edge of $\widehat{\mathcal{G}}_P$. Composing with ζ^{-1} yields maps $C : E_k \to \mathbb{N}$ and $L : E_k \to \mathbb{N}$ with $C = c \circ \zeta^{-1}$ and $L = h \circ \zeta^{-1}$.

Lemma 6.4.9 (Image is a dual hive)**.** *The resulting pair of maps (C, L) is a discrete two-colored dual hive $\zeta(P)$ in $\mathcal{H}(c(\omega), l(\omega), N)$.*

Proof. Lemma 6.3.9 and the color rules introduced before in case of blowing-up of even degree vertices yield that edges around each trivalent vertex of $\widetilde{\mathcal{G}}_P$ are colored $(0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 3)$ or $(0, 1, m)$ in the clockwise order, which translates into the same color rule around each triangular face of T_k.

It remains to prove that the map L on E_k satisfies the two conditions of Definition 6.4.1. The sum condition (1) around a triangle is a consequence of Lemma 6.3.7 in case no edge is colored m, and the direct deduction of the blowing up of vertices of even degree in case one of the edges is colored m.

The condition (2) is checked case by case. By Lemma 6.3.9 and the definition of L on edges colored 1 coming from the blowing-up of even vertices, $L(e) = L(e')$ for any opposite edges e, e' of a lozenge with middle edge colored m, yielding the condition $(2.a)$.

Without loss of generality, suppose that $e \in E_k$ has type ℓ and let s be a lozenge such that e is a border edge of type ℓ of s and the opposite edge e' is a translation of e such that $h(e) \geqslant h(e')$ and $e_{\ell+1} \leqslant e'_{\ell+1}$. Hence, either $h(e) = h(e')$, $e_{\ell+1} = e'_{\ell+1} - 1$ and the middle edge f of s is of type $k = \ell - 1$ or $h(e') = h(e) - 1$, $e'_{\ell+1} = e_{\ell+1}$ and the middle edge f is of type $k = \ell + 1$. Suppose now that at least one of the edges of the lozenge is colored m and the middle edge f is not dual to an edge of $\hat{\mathcal{G}}_P$ coming from blowing-up an even vertex. Hence, f corresponds to a strip S_f of type k. Moreover, there exist $\tilde{e}, \tilde{e'} \in E_N^2$ of type ℓ with $h(\tilde{e}) = L(e)$ and $h(\tilde{e}') = L(e')$, $\tilde{f}, \tilde{f}' \in S_f$ of type k such that $\tilde{e}$ and $\tilde{f}$ (resp. $\tilde{e}'$ and $\tilde{f}'$) comes from a same vertex v (resp. v') of P_v (either directly or after a blowing-up).

Remark that $\tilde{f}_\ell \geqslant \tilde{f}'_\ell$ for otherwise, in the strip S_f, there would be an edge of type different from f and labeled 0 or 1 coming from v, which is not possible from Figure 6.10. Then, if $\tilde{e}$ is of type ℓ and $\tilde{f}$ is of type $\ell - 1$ coming from a same triangle of T_N, we resume in Figure 6.12 the relation between $\tilde{f}_\ell$ and $h(\tilde{e})$ depending on the orientation of the triangle and the colors of the boundary edges (the color and position of $\tilde{f}$ is bold). Those relations are consequences of Lemma 6.3.9 and blowups of even degree vertices.

Direct triangle	$\mathbf{x}$ z y $\quad x,y,z \neq m$	$\mathbf{1}$ m 0	$\mathbf{0}$ 1 m	$\mathbf{m}$ 0 1
$\tilde{f}_\ell$	$h(\tilde{e})+1$	$h(\tilde{e})+2$	$h(\tilde{e})+1$	$h(\tilde{e})+1$

Reversed triangle	y z $\mathbf{x}$ $\quad x,y,z \neq m$	0 m $\mathbf{1}$	1 0 $\mathbf{m}$	m 1 $\mathbf{0}$
$\tilde{f}_\ell$	$h(\tilde{e})+1$	$h(\tilde{e})$	$h(\tilde{e})+1$	$h(\tilde{e})$

Figure 6.12: Coordinates of an edge in function of the coloring and height of the next edge in a triangle ($\tilde{f}$ correspond to the bold edge and $\tilde{e}$ corresponds to the horizontal edge).

From those relation and the fact that $\tilde{f}_\ell \geqslant \tilde{f}'_\ell$, we deduce that $h(\tilde{e}) \geqslant h(\tilde{e}')$, i.e $L(e) \geqslant L(e')$, in the case $e'_{\ell+1} = e_{\ell+1}+1$ and that $h(\tilde{e}) > h(\tilde{e}')$, i.e $L(e) > L(e')$, if one of the boundary edge of s different from e is colored m. The case $e_\ell = e'_\ell + 1$ is done similarly, yielding always $h(\tilde{e}) > h(\tilde{e}')$, i.e $L(e) > L(e')$.
Finally, if the middle edge is coming from the blowing up of an even vertex and is not colored m, then this edge is necessarily colored 1, and thus e and e' are colored 0 and the opposite edges of their lozenge are colored 1. The strict inequality is directly deduced from construction of $L = h \circ \zeta^{-1}$ and Lemma 6.3.9 giving the height of edges colored 0 in an even vertex.
Hence, T_k with the labelling (C, L) is a genuine discrete two-colored dual hive, which we denote by $\zeta(P)$. The boundary values are directly deduced from the boundary of P, so that the resulting hive is in $\mathcal{H}(c(\omega), l(\omega), N)$. $\square$

Proof of Theorem 6.4.3. Let us construct the reverse bijection. Let $H = (C, L)$ be a two-colored dual hive in $\mathcal{H}(c(\omega), l(\omega), N)$. We first define the candidate vertex partition (without the coloring for now) $\mathcal{P}_v$ as follows :

- for each triangle face $t = (t^0, t^1, t^2) \in T_k$, with t^ℓ of type ℓ, with boundary colors in $(0,0,0)$, $(1,1,1)$ or $(1,0,3)$ (up to a cyclic order), we define a block $B_t \subset E_N$ with edges e^0, e^1, e^0, with e^ℓ of type ℓ, and such that :

$$h(e^\ell) = L(t^\ell),\ e^\ell_{\ell+1} = L(t^{\ell+1}) + 1.$$

- for each long rhombus with boundary $u = (s^0, t^1, \ldots, t^r, s^1, t^{r+1}, \ldots, t^{2r})$ with s^i of type ℓ and t^i of type $\ell+1$ and $t^1_i = s^0_i + 1$ such that $C(s^0) = C(s^1) = 1$, $C(t^i) = 0$, $L(t^i) = L(t^{2r+1-i}) = L(t^1) - (i-1)$ for $1 \leqslant i \leqslant 2r$, and which is not included in an other rhombus satisfying such property, we define a block $B_u \subset E_N$ with edges $(e^0, e^1, f^1, \ldots, f^{2r})$ with $h(v) = L(v)$ for $v \in B_u$ and

$$e^0_{\ell+1} = L(t^1) + 2,\ e^1_{\ell+1} = L(t^r),\ f^i_{\ell+2} = f^{2r+1-i}_{\ell+2} - 1 = e^0_{\ell+2} + i,\ 1 \leqslant i \leqslant r.$$

  Remark that for $1 \leqslant i \leqslant r$, $f^i_{\ell+2} = f^{2r+1-i}_{\ell+2} - 1 = t(g^i)$, where g^i is the edge of type $\ell+2$ colored m adjacent to f^i or f^{2r+1-i}.

- for each edge s of type ℓ on the boundary of T_k, we define a singleton in $B_e \subset E_N$ consisting of the unique edge e of type ℓ with $h(e) = L(s)$.

Moreover, by Lemma 6.4.10 below, edges e, e' coming from different edges $t, t' \in T_k$ by the previous constructions are distinct. Let $\mathcal{E} = \bigcup_{B \in \mathcal{P}_v} B$. We can thus define a coloring c on $\mathcal{E}$ by setting $c(e) = C(t)$ when e is constructed from t above. By construction and the property (1) of Definition 6.4.1, the covering $\mathcal{P}_v$ satisfies all the properties of Lemma 6.3.9. With the above constructions and the conditions (1) and (2.a) of a two-color dual hive, we can then check that all the relations from Figure 6.12 is still satisfied when the bold edge is an element of $\mathcal{E}$.

Define then a relation $\sim$ on $\mathcal{E}$ by saying that $e \sim e'$ if e, e' are coming from a same edge of T_k through the previous construction, and denote by $\mathcal{P}_e$ the set partition coming from this relation. By the properties of a two-colored dual hive, any long rhombus considered before of border edges of type $\ell, \ell+1$ has its inner middle edges of type $\ell-1$ colored m, so that none of the triangles inside this long rhombus yields block of $\mathcal{P}_v$ through the first step. Hence, each edge of T_k yields at most 2 edges of $\mathcal{E}$. Remark that an edge of T_k is either adjacent to two faces or to one face and the boundary of T_k, so that $\mathcal{P}_e$ consists of pairs or singletons, and in the latter case the singleton belongs to two blocks of $\mathcal{P}_v$. If $e \sim e'$, by the above construction $c(e) = c(e'), L(e) = L(e')$ and e, e' have the same type, so that $\mathcal{P}_e$ only has admissible pairs (which can be reduced to a singleton). If $e \neq e'$ belong to a same block of $\mathcal{P}_v$ they come from different edges of T_k and thus $\{e, e'\} \notin \mathcal{P}_e$.

In view of applying Proposition 6.3.14, it suffices to prove that two pairs of $\mathcal{P}_e$ do not cross. Suppose that $B = \{e^1, e^2\}$ and $B' = \{e^3, e^3\}$ are two blocks of $\mathcal{P}_e$. If they are of same type ℓ, then Lemma 6.4.10 yields that $B \cap S_{B'} = B' \cap S_B = \emptyset$. If B are of different type ℓ and $\ell+1$, Lemma 6.4.11 yields that the second condition of crossing strips is never satisfied, and the first condition may only be satisfied in the case (4) of Lemma 6.4.11 where $t'_\ell \geqslant t_\ell$ and $t'_{\ell+1} \geqslant t_{\ell+1}$, when $e^2_{\ell+1} = e^3_{\ell+1}$. But in the latter case, by Definition 6.3.10, edges of type $\ell+1$ of the strip S_B have $\ell+1$-coordinate strictly smaller than $e^2_{\ell+1}$, so that $B' \cap S_B = \emptyset$. Likewise, edges of the strip $S_{B'}$ of type ℓ have $\ell+1$-coordinate strictly larger than $\min(e^3_{\ell+1}, e^4_{\ell+1})$ so that $B \cap S_{B'} = \emptyset$. Hence, S_B and $S_{B'}$ do not cross.

Pairs of $\mathcal{P}_e$ are admissible and any two different pairs $B, B' \in \mathcal{P}_e$ do not cross, thus partition $\mathcal{P}_e$ satisfies the properties of Lemma 6.3.11. Finally, by Proposition 6.3.14 applied to $(\mathcal{P}_v, \mathcal{P}_e)$, there exists a unique puzzle P such that the corresponding vertex and edge partitions are respectively $\mathcal{P}_v$ and $\mathcal{P}_e$. Denote by $\chi(H)$ this puzzle. It is clear from the above constructions that $\chi(H) \in P(\omega)$ and that $\chi \circ \zeta$ and $\zeta \circ \chi$ are respectively identity maps of $P(\omega)$ and $\mathcal{H}(c(\omega), l(\omega), N)$. $\square$

Lemma 6.4.10 (Same type blocks do not cross)**.** *Let $t \neq t' \in T_k$ of same type ℓ in blocks of $\mathcal{P}_v$, and suppose without loss of generality that $t_{\ell+1} > t'_{\ell+1}$ or $t_{\ell+1} = t'_{\ell+1}$ and $h(t') > h(t)$. Then, if $h(t) < h(t')$ and $t_{\ell+1} > t'_{\ell+1}$, the edges e^1, e^2 (resp. e^3, e^4) of T_N associated to t (resp. t') satisfy*

$$h(e^1) = h(e^2) < h(e^3) = h(e^4),$$

if $h(t) < h(t')$ and $t_{\ell+1} = t'_{\ell+1}$,

$$\min(e^1_{\ell-1}, e^2_{\ell-1}) > \max(e^3_{\ell-1}, e^4_{\ell-1})$$

and if $h(t) \geqslant h(t')$ and $t_{\ell+1} > t'_{\ell+1}$,

$$\min(e^1_{\ell+1}, e^2_{\ell+1}) > \max(e^3_{\ell+1}, e^4_{\ell+1}).$$

Proof. Since t and t' are in blocks of $\mathcal{P}_v$, neither t nor t' are colored m. If $h(t) < h(t')$ and $t_{\ell+1} > t'_{\ell+1}$, $L(t) < L(t')$ by Remark 6.4.2. Since $h(e^1) = h(e^2) = L(t)$ and $h(e^3) = h(e^4) = L(t')$, this implies

$$h(e^1) = h(e^2) < h(e^3) = h(e^4).$$

If $t_{\ell+1} = t'_{\ell+1}$ and $h(t') > h(t)$, then by Condition (2.c) of Definition 6.4.1, $L(t') > L(t)$ except if the middle edge of all lozenges between t and t' are colored m. In the latter case, let s (resp. s') be the edge of type $\ell - 1$ such that t, s form a reverse triangle (resp. t', s' form a direct triangle). Since $s_\ell = t_\ell + 1$ and $s'_\ell = t'_\ell + 1$, the inequality $t'_\ell > t_\ell$ implies that $s'_\ell > s_\ell$. Similarly, since $s_{\ell-1} = t_{\ell-1} - 1$ and $s'_{\ell-1} = t'_{\ell-1}$, we have

$$s_{\ell-1} = t_{\ell-1} - 1 = N - t_\ell - t_{\ell+1} - 1 > N - t'_\ell - t'_{\ell+1} - 1 \geqslant t'_{\ell-1} \geqslant s'_{\ell-1}.$$

Hence, $L(s') \leqslant L(s)$. Let us introduce the third edge r (resp. r') of the triangle with edges s, t (resp. s', t'). By Figure 6.12 and Condition (1) from Definition 6.4.1, we get that $e^3_{\ell-1} = N - e^3_\ell - e^3_{\ell+1} = N - L(t') - L(r') - 1 = L(s')$ and $e^2_{\ell-1} = N - e^2_\ell - e^2_{\ell+1} = N - L(t) - L(r) = L(s) + 1$ and thus $e^2_{\ell-1} = L(s) + 1 > L(s') = e^3_{\ell-1}$, so that

$$e^1_{\ell-1} \geqslant e^2_{\ell-1} > e^3_{\ell-1} \geqslant e^4_{\ell-1}.$$

If $h(t) \geqslant h(t')$ and $t_{\ell+1} > t'_{\ell+1}$, then $t'_{\ell+2} > t_{\ell+2}$. Suppose without loss of generality that $e^1_{\ell+1} \geqslant e^2_{\ell+1}$ and $e^3_{\ell+1} \geqslant e^4_{\ell+1}$. Let us consider the edges s, s' of type $\ell+1$ such that (t, s, u) and (t', s', u') are respectively direct and reverse triangles of T_k so that the corresponding edge of t and the piece containing the direct triangle is e^2 and the corresponding edge for t' and the reverse triangle is e^3. Since $h(s) = t_{\ell+1} - 1$ and $s_{\ell+2} = t_{\ell+2} + 1$ and $h(s') = t'_{\ell+1} - 1$ and $s'_{\ell+2} = t'_{\ell+2}$, $h(s) > h(s')$ and $s'_{\ell+2} \geqslant s_{\ell+2}$, so that $L(s) > L(s')$ by Remark 6.4.2. Then, since (t, u, s) is a direct triangle, Figure 6.12, $e^2_{\ell+1} = L(s) + 1$, except if $c(t) = 1, c(s) = 0$ and $c(u) = m$ where $e^2_{\ell+1} = L(s) + 2$. Likewise, since (t', s', u') is a reverse triangle, $e^3_{\ell+1} = L(s') + 1$ except if $c(t) = 1, c(s) = 0, c(u) = m$ or $c(t) = 0, c(s) = m, c(u) = 1$ where $e^3_{\ell+1} = L(s')$. Hence, in any case,

$$e^3_{\ell+1} \leqslant L(s') + 1 < L(s) + 1 \leqslant e^2_{\ell+1}$$

and

$$e^1_{\ell+1}, e^2_{\ell+1} > e^3_{\ell+1}, e^4_{\ell+1}.$$

□

Lemma 6.4.11 (Different type blocks do not cross)**.** *Let $t, t' \in T_k$ be of respective type $\ell, \ell+1$ yielding edges in $\mathcal{E}$, and denote by e^1, e^2 (resp. e^3, e^4) the edges of T_N corresponding to t (resp. t'). Then,*

1. *if $t'_\ell > t_\ell$ and $t'_{\ell+1} < t_{\ell+1}$, then*
$$e^1_\ell = e^2_\ell < e^3_\ell \leqslant e^4_\ell.$$
2. *if $t'_\ell < t_\ell$ and $t'_{\ell+1} \geqslant t_{\ell+1}$, then*
$$e^3_\ell \leqslant e^4_\ell < e^1_\ell = e^2_\ell.$$
3. *if $t'_\ell \leqslant t_\ell$ and $t'_{\ell+1} < t_{\ell+1}$,*
$$e^3_{\ell+1} \leqslant e^4_{\ell+1} < e^1_{\ell+1} \leqslant e^2_{\ell+1}.$$

4. if $t'_\ell \geqslant t_\ell$ and $t'_{\ell+1} \geqslant t_{\ell+1}$,

$$e^1_{\ell+1} \leqslant e^2_{\ell+1} \leqslant e^3_{\ell+1} = e^4_{\ell+1}.$$

Proof. The proof of the four assertions are similar.

1. Suppose that $t'_\ell > t_\ell$ and $t'_{\ell+1} < t_{\ell+1}$. Let (t', s', r') and (t', s'', r'') be the reverse and direct triangles belonging to pieces yielding respectively e^3 and e^4, with s', s'' of type ℓ. Since then $e^4_\ell \geqslant e^3_\ell$, it suffices to show that $e^3_\ell > e^2_\ell$. Since $s'_\ell = t'_\ell - 1$ and $s'_{\ell+1} = t'_{\ell+1} + 1$, we have $s'_\ell \geqslant t_\ell$ and $t_{\ell+1} \geqslant s'_{\ell+1}$, so that $L(s') \geqslant L(t)$. Since t' is of type $\ell + 1$ not and colored m and (t', s', r') is a reverse triangle, Figure 6.12 and Condition (1) from Definition 6.4.1 yield that either $e^3_{\ell+2} = L(r') + 1$ and $L(r') + L(s') + L(t') = N - 2$ or $e^3_{\ell+2} = L(r')$ and $L(r') + L(s') + L(t') = N - 1$. In any case, $e^3_{\ell+2} = N - 1 - L(t') - L(s')$, so that

$$e^3_\ell = N - e^3_{\ell+1} - e^3_{\ell+2} = N - L(t') - (N - 1 - L(t') - L(s')) = L(s') + 1 > L(t) = e^2_\ell = e^1_\ell.$$

2. Suppose that $t'_\ell < t_\ell$ and $t'_{\ell+1} \geqslant t_{\ell+1}$, and let s' be the edge of type ℓ such that (t', s', r') is a direct triangle. Since $s'_\ell = t'_\ell$ and $s'_{\ell+1} = t'_{\ell+1} + 1$, $s'_\ell < t_\ell$ and $s'_{\ell+1} > t_{\ell+1}$, so that $L(t) > L(s')$ by Remark 6.4.2. Since (t', s', r') is a direct triangle, Figure 6.12 and Condition (1) from Definition 6.4.1 yield by a same reasoning as above that $e^4_{\ell+2} \geqslant L(r') + 1 = N - L(s') - L(t')$, so that, using that $e^4_{\ell+1} = L(t')$,

$$e^4_\ell = N - e^4_{\ell+1} - e^4_{\ell+2} \leqslant L(s') < L(t) = e^1_\ell = e^2_\ell.$$

3. Suppose that $t'_\ell \leqslant t_\ell$ and $t'_{\ell+1} < t_{\ell+1}$, and let s be the edge of type $\ell + 1$ such that (t, s) is part of a direct triangle. Then, $s_\ell = t_\ell$ and $s_{\ell+1} = t_{\ell+1} - 1$, so that $s_\ell \geqslant t'_\ell$ and $s_{\ell+1} \geqslant t'_{\ell+1}$. We deduce that $s_{\ell+2} \leqslant t'_{\ell+2}$, and thus $L(s) \geqslant L(t')$. Since $e^1_{\ell+1} \geqslant L(s) + 1$ by Figure 6.12, we thus have

$$e^3_{\ell+1} = e^4_{\ell+1} = L(t') \leqslant L(s) < e^1_{\ell+1} \leqslant e_{\ell+1}.$$

4. Suppose that $t'_\ell \geqslant t_\ell$ and $t'_{\ell+1} \geqslant t_{\ell+1}$, and let s be the edge of type $\ell + 1$ such that (t, s, r) is a reverse triangle. Then, $s_\ell = t_\ell + 1$ and $s_{\ell+1} = t_{\ell+1} - 1$. Hence, $t'_{\ell+1} > s_{\ell+1}$ and $t'_{\ell+2} = N - t'_{\ell+1} - t'_\ell \leqslant N - s_{\ell+1} - 1 - s_\ell + 1 \leqslant s_{\ell+2}$ and the inequality is strict except when $t'_\ell = t_\ell$ and $t'_{\ell+1} = s_{\ell+1}$. Hence, by Remark 6.4.2 in the case of strict inequality and Condition (2.b) and (2.c) from Definition 6.4.1, $L(t') > L(s)$, except when $t'_\ell = t_\ell, t'_{\ell+1} = s_{\ell+1}$ and $C(r) = m$, in which case $L(t') = L(s)$. In the first case, by Figure 6.12 we have $e^2_{\ell+1} \leqslant L(s) + 1 \leqslant L(t')$. In the second case, since $C(r) = m$ we have $e^2_{\ell+1} = L(s) \leqslant L(t')$, so that in any case

$$e^1_{\ell+1} \leqslant e^2_{\ell+1} \leqslant e^3_{\ell+1} = e^4_{\ell+1}.$$

□

6.5 Color swap

The goal of this combinatorial section is to exhibit a convex body of dimension $D = \frac{(n-1)(n-2)}{2}$ having integer points counted by quantum Littlewood-Richardson coefficients, so that the limit expression (6.2.26) converges to the volume of a polytope. In Section

6.5.1, we present local configurations in color maps called arrows and hexagons. Section 6.5.2 gives a propagation algorithm for another configuration called a gash. Using these propagations, we show in Sections 6.5.3 and 6.5.4 that any color map can be reduced to a simple color map as in Figure 6.20. Section 6.5.5 then extends the reduction defined on color maps to quasi dual hives, which are more general than dual hives of Definition 6.4.1.

From this section to the end of the chapter, we set $k = n + d$ and $\xi = \mathrm{e}^{\frac{i\pi}{3}}$. We also assume that the color maps of dual hives are regular in the sense of Definition 6.5.1.

Definition 6.5.1 (Regular boundaries)**.** A color map $C : E_k \to \{0, 1, 3, m\}$ is *regular*, or has *regular boundaries*, if for every $i \in \{0, 1, 2\}$,

$$c^{(i)} = (\underbrace{1, \ldots, 1}_{d \text{ times}}, \underbrace{0, \ldots, 0}_{n-d \text{ times}}, \underbrace{1, \ldots, 1}_{d \text{ times}}). \tag{6.5.1}$$

Moreover, we say that a dual hive $H = (C, L)$ is *regular* if its color map C is.

6.5.1 Arrows and hexagons

Let us start with some definitions on local configurations of edges in E_k.

Definition 6.5.2 (Opening)**.** Let $x \in T_n$. An *opening* of type $l \in \{0, 1, 2\}$ at x is a pair of edges $(e, e') \in E_n^2$ such that if $e = (e_1, e_2)$, $e' = (e'_1, e'_2)$ with $(e_1, e'_1, e_2, e'_2) \in T_n^4$ and $t(e), t(e')$ are the types of e and e',

$$e_i = e'_i = x \text{ for some } i \in \{1, 2\},$$
$$\{t(e), t(e')\} = \{l - 1, l + 1\} \text{ and } C(e) = C(e') \in \{0, 1\}.$$

The *color* of the opening is defined as the color of edges e and e'.

Consider an opening $a = (e, e')$ at x of type l and color $c \in \{0, 1\}$. Let $e'' = e''(a)$ be the edge such that e, e' are edges of the lozenge with middle edge e''. The only possible colors of the edge e'' are $C(e'') \in \{0, 1\}$. If $C(e'') = c$, the two triangular faces of the lozenge with middle edge e'' have all of their edges colored c. If $C(e'') \neq c$, then there is an opening a' of type l and color c at the other endpoint of e''. Note that there can only be finitely many such openings before $C(e'') = c$.

Definition 6.5.3 (Arrow)**.** Let $a = (e, e')$ be an opening of type l and color c. Let $r \geqslant 0$ be the number of successive openings having middle edge e'' such that $C(e'') \neq c$ with $C(e'') \in \{0, 1\}$ as in the previous paragraph. An *arrow* of length $r \geqslant 0$ at the opening a is the configuration of edges consisting of the $r \geqslant 0$ successive pairs of 3 and m lozenges together with the pair of direct and reverse faces with boundary edges of color c.

See Figure 6.13 for examples of openings and arrows.

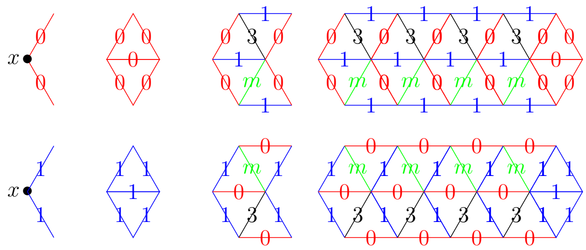

Figure 6.13: First row from left to right : an opening a with color 0 and type 0 at x, the case $C(e'') = c$, the case $C(e'') \neq c$ and an arrow of length $r = 4$. The second row is the analog for color 1.

Definition 6.5.4 (ABC hexagons)**.** Let C be a color map and let h be a hexagon, that is, the union of six triangular faces sharing one vertex in T_k. We say that h is an *ABC hexagon* (for the color map C) if the color map C restricted to h is any of the three configurations in Figure 6.14 up to a rotation.
A *rotation* of an ABC hexagon h is the replacement of the values of C by the ones obtained from a rotation of h which preserves the value of C on the boundary $\partial h \cap E_k$.

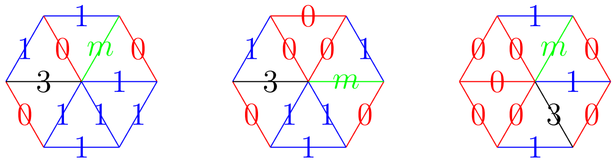

Figure 6.14: The three types of hexagons : A (left), B (center) and C (right).

Note that type B has three possible rotations whereas A and C only have two. For a hexagon h, denote E_h the edges of E_k which are in h.

Let $\mathcal{A}$ be an arrow of length $r \geqslant 1$ and type l at an opening with center x. The center of the last opening is $y \in \{x + r\xi^l, x - r\xi^l\}$. Notice that if the color c of $\mathcal{A}$ is 0 (resp. 1), then the hexagon $h(y)$ with center y is of type C (resp. A). Applying a rotation to $h(y)$ yields an arrow $\mathcal{A}'$ of length $r-1$ of type l at the same opening with center x. By applying hexagon rotations to $x + r\xi^l, x + (r-1)\xi^l, \ldots, x + \xi^l$ (or $x - r\xi^l, x - (r-1)\xi^l, \ldots, x - \xi^l$) in this order, one gets an arrow $R(\mathcal{A})$ of length r of type l the same opening with center $x + r\xi^l$. We call this sequence of r hexagon rotations the *reversal* of the arrow $\mathcal{A}$. An example of arrow reversal is given in Figure 6.15.

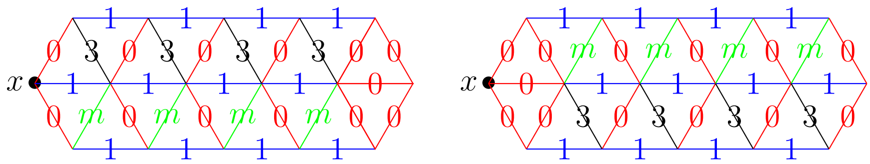

Figure 6.15: Reversal of an arrow of length 4 at x.

6.5.2 Gash propagation

We define a local configuration called a gash in Definition 6.5.5 which one can propagate in a dual hive. Local propagation rules are given in Definition 6.5.6 and the general propagation algorithm on dual hives is defined in Definition 6.5.7. The goal of the propagation algorithm is to find rigid lozenges of a dual hive in view of the next section.

Definition 6.5.5 (Gash)**.** Let $x \in T_k$. A gash g with center $x = x(g)$ is the union of the two edges $(x, x - \xi^{2l}), (x + \xi^{2l}, x)$ for $l \in \{0, 1, 2\}$ such that

$$C((x, x - \xi^{2l})) = 1,\ C((x + \xi^{2l}, x)) = 0 \text{ if } l \in \{0, 1\}$$
$$C((x, x - \xi^{2l})) = 0,\ C((x + \xi^{2l}, x)) = 1 \text{ if } l = 2.$$

The type of a gash denoted $t(g)$ is defined as the type $l \in \{0, 1, 2\}$ of its edges.

Note that this definition only depends on the color map C of H. Let g be a gash. There are only six possible configurations given in Figure 6.16 adjacent to g. In this section, we show that such a gash g can be moved across the color map C using local moves until reaching configuration (v) or (vi) of Figure 6.16.

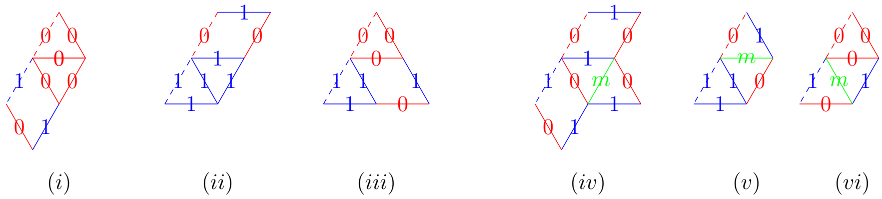

Figure 6.16: The six possible adjacent configurations to a gash of type 2 shown in dashed edges. The same holds for a gash of other types up to rotations.

Definition 6.5.6 (Gash propagation)**.** Let g be gash of type l with center x.

1. Suppose that g is adjacent to a configuration (i). Let $y = x + \xi^4$, (resp. $y = x + 1$, $y = x + \xi^5$) if $l = 0$, (resp. $l = 1, l = 2$). We call the propagation of g the gash g' of type l at center y.
2. Suppose that g is adjacent to a configuration (ii). Let $y = x + \xi^5$, (resp. $y = x + \xi, y = x + 1$) if $l = 0$, (resp. $l = 1, l = 2$). We call the propagation of g the gash g' of type l at center y.
3. Suppose that g is adjacent to a configuration (iii). Let $y = x + 1$ if $l = 1$ or $l = 2$. We call the propagation of g the gash g' of type $3 - l$ at center y.
4. Suppose that g is adjacent to a configuration (iv). Notice that there is a 0 opening at x and thus an arrow of color 0 at x with type $l + 1$. Reverting this arrow yields a configuration (i) adjacent to g and we define the propagation of g to be the gash g' of type l as in step (1). See Figure 6.17 for an illustration.

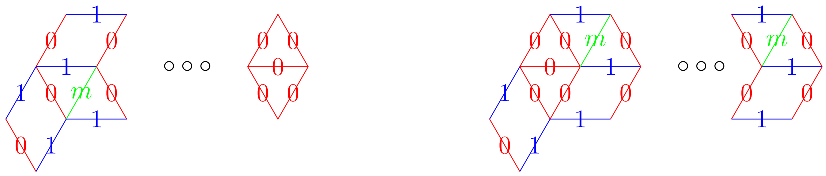

Figure 6.17: Propagation of a gash g adjacent to a configuration (iv). The arrow of color 0 has been reversed yielding a configuration (i) adjacent to g.

Remark that the only propagation in which the type of the gash changes is (iii). We now give a general procedure using local propagations from Definition 6.5.6. This procedure starts from a gash and propagates it until reaching a configuration that is either (v) or (vi).

Definition 6.5.7 (Propagation algorithm)**.** The *propagation algorithm* is the following algorithm.
Input: A color map C and a gash g of type $l \in \{1, 2\}$.

1. Set $g^{(0)} = g$, $x^{(0)} = x(g)$, $t^{(0)} = t(g)$.

2. **WHILE** $g^{(s)}$ is adjacent to $(i), (ii), (iii)$ or (iv): set $g^{(s+1)}$ to be the propagation of $g^{(s)}$ with center $x^{(s+1)}$ and type $t^{(s+1)}$.

Proposition 6.5.8 (Propagation algorithm is correct)**.** *Let g be a gash of type 2 in T_k. The propagation algorithm terminates at a gash $\tilde{g}$ adjacent to configuration of type (v) or (vi).*

For the proof of Proposition 6.5.8, we need Lemma 6.5.9 which shows that any triangular region having two of its sides with edges colored $c \in \{0, 1\}$ has all its edges colored c.

Lemma 6.5.9 (Regular equilateral triangles)**.** *Let $C : E_k \to \{0, 1, 3, m\}$ be a color map on edges of T_k. Let R be any subset of edges of E_k such that ∂R is an equilateral triangle of size $s \geqslant 1$. Let $c \in \{0, 1\}$ and assume that two boundaries of R have edges e which are all colored c. Then, every edge in R is colored c.*

Proof. Let us first show a general fact about a shape described below that we call a cup. For $r \geqslant 1$, we call a cup of length r and type i the union of r consecutive type i edges together with one edge of type $i+1$, respectively of type $i-1$, forming an angle of $\frac{2\pi}{3}$ with type i edge with maximal and minimal heights. See Figure 6.18 for an example. Suppose that edges of a cup are all colored $c \in \{0, 1\}$. Let us show by induction on r that the only possible color of edges in the convex hull of the cup is c.

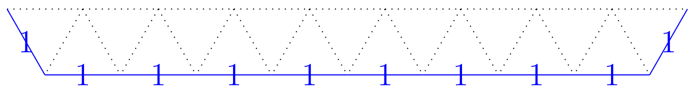

Figure 6.18: A cup of length $r = 8$ and type 0 with color $c = 1$. Edges in the convex hull of the cup are dotted

Two adjacent edges of the cup of different types form an opening of color c. One checks that the arrow at this opening has length zero for otherwise all the r edges of type i would belong to some non-rigid lozenges with opposite edges of type i (resp. $i-1$) colored c (resp. $1-c$) which is incompatible with the color c of the other opening of the cup. Thus, the arrow has zero length. Then, the rest of the convex hull is a cup of length $r-1$ which completes the proof of cup completion by induction.

Let us consider the region R of the statement, and suppose that the boundary of type $i-1$ and $i+1$ of R are colored c. The corner of the triangular face between boundaries of type $i-1$ and $i+1$ has all of its edges colored c, as two of them lie on the boundary of R. This induces a cup of length 1 and type i having edges of color c. By the previous reasoning, its only color completion consists of edges of color c. Each completion of a cup of size r with $r < s$ yields a cup of size $r+1$ with edges colored c. Filling cups of sizes $1, 2, \ldots, s-1$ with edges of color c fills then R with edges colored c and thus proves the claim. □

Corollary 6.5.10 (Corners of regular boundaries)**.** *Let $C : E_k \to \{0, 1, 3, m\}$ be a regular color map. Then, the three equilateral regions of side length d each containing an extremal vertex of T_k have all of their edges colored 1.*

Proof. Notice that the regularity of the color map C implies that each such triangular region of side length d has two boundaries which lie on $\partial T_k^{(0)} \cup \partial T_k^{(1)} \cup \partial T_k^{(2)}$ having edges colored 1. Applying Lemma 6.5.9 yields the result. □

Proof of Proposition 6.5.8. At each step of the gash propagation, one has that $x_0^{(s+1)} < x_0^{(s)}$ or $x_1^{(s+1)} > x_1^{(s)}$ and $t^{(s)} \in \{1,2\}$ which implies that the while loop terminates on a last gash $g^{(\infty)}$. Since $g^{(\infty)}$ is the last gash, it is of type 1 with center $x^{(\infty)} \in \partial T_k^{(1)}$. As the boundary $\partial T_k^{(1)}$ is regular, the edge of color 1 in $g^{(\infty)}$ is the edge with height n so that $x_1^{(\infty)} = n$, see Figure 6.19. Moreover, by Corollary 6.5.10, the equilateral triangle $R_d \subset E_k$ of length d having one of its boundaries between $x^{(\infty)}$ and $(n,0)$ has all of its edges colored 1.

Assume for the sake of contradiction that $g^{(\infty)}$ is not adjacent to a configuration (v) or (vi). Consider the last (iii) configuration encountered before reaching $\partial T_k^{(1)}$ having a gash g' of type 1. Note that such a configuration exists as all other configurations preserve the type of the gash during propagation of Definition 6.5.6. Consider the last gash g with center x resulting from a propagation of type (i) after g' with the convention that $g = g'$ if no configuration (i) or (iv) happen after g' (recall that configuration (iv) reduces to a propagation of type (i), see step (4) of Definition 6.5.6). Propagations $(g^{(s)}, s \geqslant 0)$ after $g = g^{(0)}$ are then of type (ii) so that for $s \geqslant 0$, $x_1^{(s+1)} = x_1^{(s)}$ and $x_0^{(s+1)} = x_0^{(s)} - 1$. Since $x^{(\infty)} \in R_d$, we would have $x \in R_d$ and the edge $e' \in E_k$ with origin $x + \xi^5$ of type 0 and color 0 in the last (i) configuration before g (or in the last configuration (iii) if no such configuration (i) exists) would have both its endpoints in R_d which contradicts the fact that edges in R_d are all colored 1. See Figure 6.19 for an illustration of the above argument. □

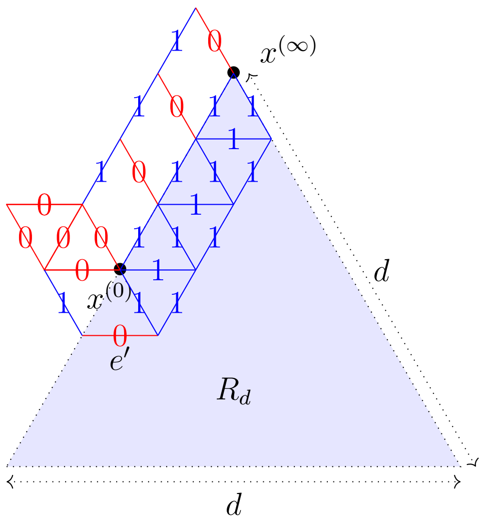

Figure 6.19: The 0 colored edge e' in the last configuration (i) would be in R_d.

Remark 6.5.11 (Type 1 propagation algorithm)**.** Note that the argument given in the proof of Proposition 6.5.8 remains valid in the case of a type 1 gash g with center x such that the type 0 edge with origin $x + \xi^5$ is colored 0.

6.5.3 Color swap path

In Proposition 6.5.8, we showed that any gash g of type 2 can be propagated to find a configuration (v) or (vi) having a rigid lozenge in it. In this section, we show that via hexagon rotations of Definition 6.5.4, one can bring this rigid lozenge at the location of $x = x(g)$.

Lemma 6.5.12 (Back propagation)**.** *Let g be gash with center x adjacent to a configuration (i, ii) or (iii) and suppose that its propagation g' with center x' is adjacent to a configuration (v) or (vi). Then, the hexagon h' with center x' is an ABC hexagon. Moreover, using a hexagon rotation of h', the hexagon h with center x is an ABC hexagon.*

Proof. One can check that the statement of Lemma 6.5.12 holds in all possible cases of configurations, see Figure 6.14. □

Proposition 6.5.13 (Gash reduction)**.** *Let C be a color map and let g be a gash of type 2 in C. Let $\tilde{g} = g^{(s)}$ for some $s \geqslant 0$ be the last gash as in Proposition 6.5.8. Using hexagon rotations of ABC hexagons with centers given by $x^{(s)}, x^{(s-1)}, \ldots, x^{(0)}$ in this order, C can be mapped to a color map C' such that g is a gash for C' adjacent to a configuration (v) or (vi).*

Proof. Applying Lemma 6.5.12 to every center $x^{(s)}, x^{(s-1)}, \ldots, x^{(1)}$ in this order yields the desired configuration. □

Remark 6.5.14 (Reduction of a type 1 gash)**.** As in Remark 6.5.11, the result of Proposition 6.5.13 still holds if one considers g of type 1 with center x such that the type 0 edge with origin $x + \xi^5$ is colored 0.

6.5.4 Color map reduction

Recall that we only consider regular boundary conditions for T_k that is, the color map $C : E_k \to \{0, 1, 3, m\}$ is given by $1 \ldots 10 \ldots 01 \ldots 1$ on every boundary of T_k, where there are d ones on each side of the $n - d$ zeros. The goal of this section is to show that any regular color map can be mapped via hexagon rotations to the simple color map of Definition 6.5.15.

Definition 6.5.15 (Simple color map)**.** Let $n \geqslant d \geqslant 0$ and $k = n + d$. The color map $C_0 : E_k \to \{0, 1, 3, m\}$ called the simple color map is defined by

1. C_0 is regular and thus $C_0(e) = 1$ for every edge $e \in E_k$ in any corner equilateral triangle of side length d in T_k as in Corollary 6.5.10,

2. $C_0(e) = 1$ for every edge $e \in E_k$ in the lozenge of side length d in T_k having outer vertices $n\xi, n\xi + d\xi, n\xi + d, n\xi + d\xi^5$,

3. $C_0(e) = 0$ for every edge $e \in E_k$ in the equilateral triangle having outer vertices d, n and $d + n\xi$,

4. $C_0(e) = m$, respectively $C_0(e) = 3$ for every edge $e \in E_k$ of type 0 with origin x such that $d \leqslant x_0 \leqslant n - 1$ and $1 \leqslant x_1 \leqslant d$, respectively for every edge $e \in E_k$ of type 0 with origin y such that $0 \leqslant y_0 \leqslant d - 1$ and $d + 1 \leqslant y_1 \leqslant n$.

Figure 6.20 shows an example of the simple color map C_0 for $k = 5$ and $d = 2$.

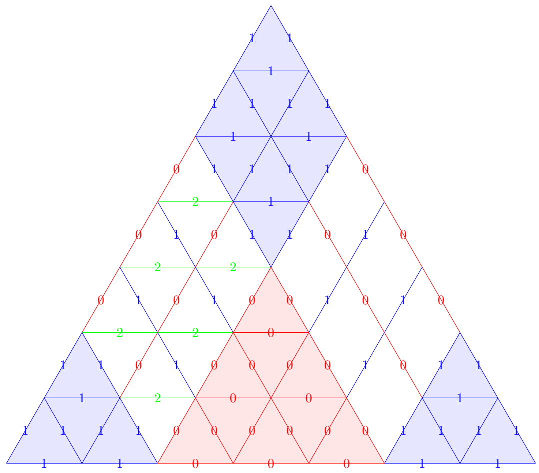

Figure 6.20: The color map C_0 for $k = 5$ and $d = 2$. The label 2 corresponds to the label m. Uncolored edges have color 3 (picture done with the module *Knutson-Tao puzzles* of Sage [The20]).

Proposition 6.5.16 (Color map reduction)**.** *Let $C : E_k \to \{0, 1, 3, m\}$ be a regular color map. Using hexagon rotations, one can map C to C_0, where C_0 is the simple color map of Definition 6.5.15.*

Proof. Let $\{x^{(s)}, 1 \leqslant s \leqslant d(k-d)\}$ be the vertices in T_k ordered such that for $1 \leqslant s \leqslant d(k-d)$, $s - 1 = s_1 d + s_2$ with $s_1 \geqslant 0$ and $0 \leqslant s_2 \leqslant d-1$,

$$x^{(s)} = (d + s_1)\xi + s_2\xi^5.$$

Let C be a regular color map on E_k. Let us show by induction on s that using hexagon rotations, C can be mapped to a regular color map $C^{(s)}$ such that the type 0 edges with origins $x^{(1)} + 1, \dots, x^{(s)} + 1$ are colored m. We first prove it for $s = 1$. Notice that since C is regular, there is a gash $g^{(1)}$ of type 2 with center $x^{(1)}$. Applying Proposition 6.5.13 yields that using hexagon rotations, C can be mapped to $C^{(1)}$ such that $g^{(1)}$ is adjacent to a configuration (v) or (vi). Since $C^{(1)}$ is regular, this configuration is necessarily (v) which implies that the edge of type 0 with origin $x^{(1)} + 1$ is colored m.
Assume that $C^{(s)}$ is a color map such that the type 0 edges with origins $x^{(1)}+1, \dots, x^{(s)}+1$ are colored m. Notice that there is a 01 opening at $x^{(s+1)}$ that is, the edges $(x^{(s+1)}, x^{(s+1)} + \xi)$ and $(x^{(s+1)}, x^{(s+1)} + \xi^5)$ are colored respectively 0 and 1. A 01 opening has only three possible completions showed in Figure 6.21.

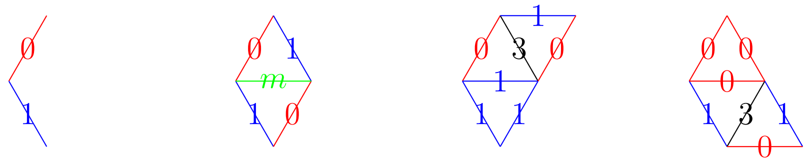

Figure 6.21: A 01 opening (left) and its possible completions

In the case of the first completion, the color map $C^{(s+1)} = C^{(s)}$ satisfies the desired conditions. In the case of the third and fourth completion, there is a gash $g^{(s+1)}$ of type 2 and 1 respectively with center $x^{(s+1)} + 1$. Applying Proposition 6.5.13 and Remark 6.5.14 respectively shows that using hexagon rotations along the propagation path started from

$g^{(s+1)}$, the latter is adjacent to a configuration (v) or (vi). Note that rotations in this propagation path do not affect edges colored m with origins $x^{(1)}+1,\dots,x^{(s)}+1$ thanks to the above ordering. Using a hexagon rotation for the hexagon with center $x^{(s+1)}+1$ yields a color map $C^{(s+1)}$ such that the type 0 edges with origins $x^{(0)}+1,\dots,x^{(s+1)}+1$ are colored m and ends the induction. Therefore, C is equivalent up to hexagon rotations to the color map where type 0 edges with origins $x^{(0)}+1,\dots,x^{(d(k-d))}+1$ are colored m. From this configuration, there is only one possible color map to complete the rest of the hive E_k which is the simple color map C_0. □

Since the number of pieces of each type is preserved under hexagon rotations, one derives the following enumerations (only the enumeration of edges colored m will be used afterwards).

Corollary 6.5.17 (Tiles enumeration)**.** *Let H be a dual hive with regular boundaries associated to integers n and d. Let $hc(H)$, respectively, $sc(H)$, be the number of m, respectively 3 colored edges inside H. Then,*

$$hc(H) = d(n-d) = sc(H). \tag{6.5.2}$$

Moreover, for $i \in \{0,1\}$, denote $nt^{(i)}(H)$, respectively $st^{(i)}(H)$, the number of direct, respectively reversed, triangular pieces of size 1 with color i on each side. Then,

$$nt^{(0)}(H) = \frac{(n-d)(n-d+1)}{2},\ st^{(0)}(H) = \frac{(n-d)(n-d-1)}{2},$$
$$nt^{(1)}(H) = d(2d+1) \text{ and } st^{(1)}(H) = d(2d-1).$$

6.5.5 Quasi dual hives

The goal of this section is to extend hexagon rotations to hives. As of now, hexagon rotations map one color map to another. To also change label maps, we need to relax the inequality constraints of Definition 6.4.1 leading to quasi hives of Definition 6.5.20. From this section to the end, we view regular dual hives of T_{n+d} as in the discrete hexagon $R_{d,n}$, see Definitions 6.5.18, 6.5.19 and Figure 6.22 below.

Definition 6.5.18 (Hexagonal dual hives)**.** Let $n, d \geqslant 1$. Denote $E_{n,d} = \{\{u,v\} \in R_{d,n}^2 \mid d(u,v) = 1\}$ the set of edges of the discrete hexagon $R_{d,n}$. A *hexagon dual hive* is a pair of maps (C,L), $C : E_{n,d} \to \{0,1,3,m\}$ and $L : E_{n,d} \to \frac{1}{N}\mathbb{Z}$ such that $NL(.)$ satisfies the conditions of Definition 6.4.1 restricted to edges $e \in E_{n,d}$.

Definition 6.5.19 (Boundary value of a hexagonal dual hives)**.** Define the following subsets of $E_{n,d}$ for $l \in \{0,1,2\}$.

$$\partial^{(l,l)}E_{n,d} \coloneqq \{e \in E_{n,d} \mid e \text{ is of type } l \text{ and } e_{l-1} = 0\}$$
$$\partial^{(l,l+1)}E_{n,d} \coloneqq \{e \in E_{n,d} \mid e \text{ is of type } l \text{ and } e_{l-1} = n\}.$$

The *boundary value*

$$[(c^{(0,0)}, c^{(1,1)}, c^{(2,2)}, c^{(0,1)}, c^{(1,2)}, c^{(2,0)}), (\ell^{(0,0)}, \ell^{(1,1)}, \ell^{(2,2)}, \ell^{(0,1)}, \ell^{(1,2)}, \ell^{(2,0)})]$$

of a hexagonal dual hive is the restriction of (C,L) to $\partial E_{n,d}$ where

- $c^{(l,l)} \in \{0,1,3,m\}^{n-d}$ (resp. $c^{(l,l+1)} \in \{0,1,3,m\}^{d}$) is the restriction of C to $\partial^{(l,l)}E_{n,d}$ (resp. $\partial^{(l,l+1)}E_{n,d}$),

- $\ell^{(l,l)} \in (\frac{1}{N}\mathbb{N})^d$ (resp. $\ell^{(l,l+1)} \in (\frac{1}{N}\mathbb{N})^d$) is the restriction of $L(.)$ to $\partial^{(l,l)}E_{n,d}$ (resp. $\partial^{(l,l+1)}E_{n,d}$).

For $(c,l) = (c^{(0,0)}, c^{(1,1)}, c^{(2,2)}, c^{(0,1)}, c^{(1,2)}, c^{(2,0)}, \ell^{(0,0)}, \ell^{(1,1)}, \ell^{(2,2)}, \ell^{(0,1)}, \ell^{(1,2)}, \ell^{(2,0)})$ an element in $\{0,1\}^{3(n-d)} \times \{0,1\}^{3d} \times (\frac{1}{N}\mathbb{N})^{3(n-d)} \times (\frac{1}{N}\mathbb{N})^{3d}$, we denote by $\mathcal{H}_{hex}(c,l)$ the set of hexagonal dual hives with boundary value (c,l).

Let (λ,μ,ν) be partitions of length n with first part smaller than $N-n$ such that $|\lambda|+|\mu| = |\nu| + Nd$ and such that $\min(\lambda_n, \mu_n, N-n-\nu_1) \geqslant d-1$. Then, the associated boundary labels $(l^{(0)}, l^{(1)}, l^{(2)})$ defined in Definition 6.4.1 on T_{n+d} are given by

$$\begin{aligned} l^{(0)} &= (0,\ldots,d-1, N-n-\nu_1, N-n-\nu_2+1,\ldots,\\ &\quad N-\nu_{n-d}-d-1, N-\nu_{n-d+1}-d,\ldots,N-\nu_n-1)\\ l^{(1)} &= (0,\ldots,d-1,\mu_n,\mu_{n-1}+1,\ldots,\mu_{d+1}+n-d-1,\mu_d+n-d,\ldots,\mu_1+n-1)\\ l^{(2)} &= (0,\ldots,d-1,\lambda_n,\lambda_{n-1}+1,\ldots,\lambda_{d+1}+n-d-1,\lambda_d+n-d,\ldots,\lambda_1+n-1) \end{aligned}$$

so that the associated boundary colors $(c^{(0)}, c^{(1)}, c^{(2)})$ are regular in the sense of Definition 6.5.1 which means that on each boundary of T_{n+d}, the d first and last edges are colored 1 and the remaining $n-d$ edges are colored 0. By Lemma 6.5.10, any dual hive $H = (C,L) \in H(\lambda,\mu,\nu,N)$ with regular boundary conditions has every of its equilateral triangles of size d anchored in a corner of T_{n+d} colored 1. Each of these triangular regions have one boundary with labels equal to $(0,\ldots,d-1)$. The corresponding region in the puzzle yields a unique position of the corresponding triangular pieces with edge colors $(1,1,1)$ and this unique configuration gives labels to the third boundary edge of the region. Let $\tilde{l}^{(0)}, \tilde{l}^{(1)}, \tilde{l}^{(2)}$ be the respective labels of edges in $\partial^{(0,1)}E_{n,d}, \partial^{(1,2)}E_{n,d}, \partial^{(2,0)}E_{n,d}$. Then, reading decreasingly with respect to edges heights $h(e)$, these labels are given by the following (see Figure 6.22 below),

$$\tilde{l}^{(0)} = (N-n-\lambda_d+d-1, N-n-\lambda_{d+1}+d,\ldots,N-n-\lambda_1), \tag{6.5.3}$$
$$\tilde{l}^{(1)} = (N-n-\mu_d+d-1, N-n-\mu_{d+1}+d,\ldots,N-n-\mu_1), \tag{6.5.4}$$
$$\tilde{l}^{(2)} = (\nu_{n-d+1}+d-1,\ldots,\nu_{n-1}+1,\nu_n). \tag{6.5.5}$$

In this section, we now view regular dual hives $H = (C,L)$ on T_{n+d} as hexagonal dual hives by restricting C and L to $E_{n,d}$. By the above, the boundary conditions on $E_{n,d}$ are given by

$$c^{(l,l)} = (0,\ldots,0),\ c^{(l,l+1)} = (1,\ldots,1),\ \ell^{(l,l)} = \frac{1}{N}L_{|\partial^{(l,l)}E_{n,d}},\ \ell^{(l,l+1)} = \frac{1}{N}\tilde{l}^{(l)}. \tag{6.5.6}$$

for $l \in \{0,1,2\}$. We write $H_{hex}(\lambda,\mu,\nu,N)$ for the set of hexagonal dual hives having boundary colors and labels given by (6.5.6) coming from restriction of dual hives on T_{n+d}. This restriction

$$H(\lambda,\mu,\nu,N) \to H_{hex}(\lambda,\mu,\nu,N) \tag{6.5.7}$$

is a bijection where the inverse map is given by extending C from the hexagon $R_{d,n}$ to the triangle T_{n+d} setting $C(e) = 1$ for $e \in E_{n+d} \backslash E_{n,d}$ and completing the labels L in the unique possible way in corner triangles. In this section, we now view dual hives in the hexagon $R_{d,n}$ and we will omit the subscript writing $H(\lambda,\mu,\nu,N)$ for notation convenience.

Definition 6.5.20 (Quasi label map, quasi dual hive)**.** Let $C : E_{n,d} \to \{0,1,3,m\}$ be a color map and $N \geqslant 1$. A *quasi label map* is a map $L : E_{n,d} \to \frac{1}{N}\mathbb{Z}$ such that $NL(.)$

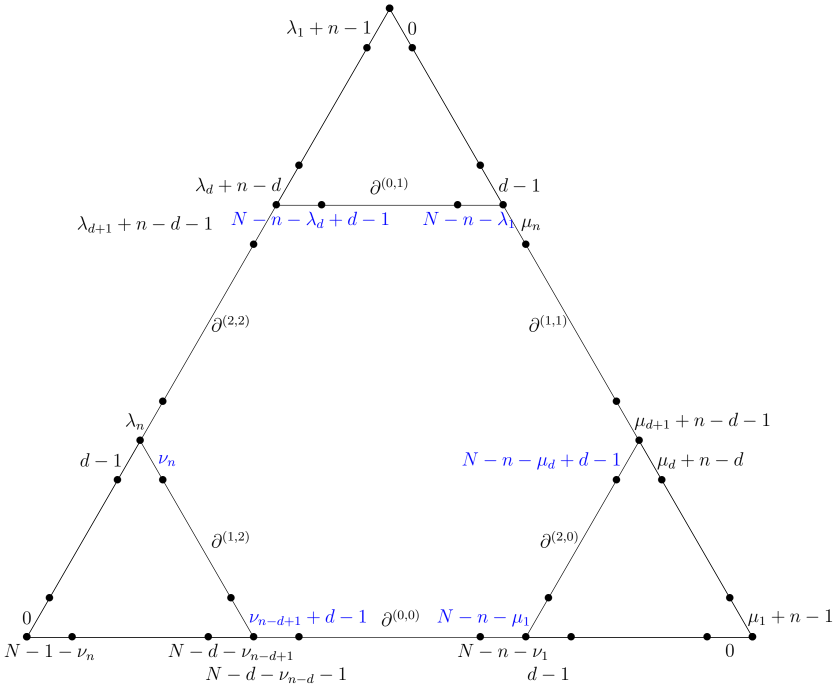

Figure 6.22: The restriction of a regular dual hive on E_{n+d} to the hexagon $E_{n,d}$ and induced boundary value for the label map. Blue labels are determined by the unique label map on each triangular corner. Multiplying edge labels by $\frac{1}{N}$ yields boundary conditions in $H_{hex}(\lambda, \mu, \nu, N)$.

satisfying the equality conditions of Definition 6.4.1, that is, equality condition on every triangular face and rigid lozenges with respect to the color map C, and boundary values on $\partial E_{n,d}$ given by the corresponding two-colored dual hives. Denote $\widetilde{L}^C(\lambda, \mu, \nu, N)$ the set of such label maps.
A *quasi dual hive* is the data of a color map C and a quasi label map L. We denote by $\widetilde{H}(\lambda, \mu, \nu, N)$ the set of quasi dual hives with boundary conditions (λ, μ, ν).

The difference with label maps of dual hives is that one does not impose the inequality constraints of Definition 6.4.1 inside the hexagonal region $E_{n,d}$. Note that dual two-colored hives are in particular quasi dual hives that is, $H(\lambda, \mu, \nu, N) \subset \widetilde{H}(\lambda, \mu, \nu, N)$.

Lemma 6.5.21 (Boundary value determine interior)**.** *Let $H = (C, L)$ be a quasi dual hive and h an ABC hexagon for its color map C. Then, the values of L on E_h°, the set of interior edges of h, are uniquely determined by the values of L on boundary edges of h and by the position of the rigid lozenge in h. Moreover, the values of L on E_h° are affine combinations of the values of L on boundary edges of h.*

Proof. Suppose that the values of L on ∂h are given by $l_1, \dots, l_6$ and values on E_h° by $l_7, \dots, l_{12}$. We will do the proof for a type A hexagon and the other types B and C can be treated using similar arguments. Without loss of generality up to some permutation of the indexes suppose that $C(e_7) = m$, see Figure 6.23.

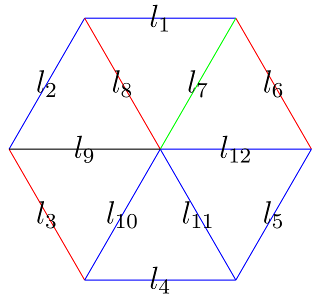

Figure 6.23: Labels of edges in h a type A hexagon.

By the equality conditions on opposite sides of the rigid lozenge, one has $l_{12} = l_1$ and $l_8 = l_6$. Let us show that the values l_7, l_9, l_{10}, l_{11} are uniquely determined in the region. Writing equality conditions on the five triangular faces gives

$$\begin{aligned} l_9 + l_8 + l_2 &= 1 - \frac{1}{N} \\ l_{10} + l_9 + l_3 &= 1 - \frac{2}{N} \\ l_{11} + l_{10} + l_4 &= 1 - \frac{1}{N} \\ l_{12} + l_{11} + l_5 &= 1 - \frac{2}{N} \\ l_{12} + l_7 + l_6 &= 1 - \frac{1}{N} \end{aligned}$$

which implies

$$l_1 + l_3 + l_5 = l_2 + l_4 + l_6 - \frac{2}{N}. \tag{6.5.8}$$

Suppose that the former holds. The fourth equations of the system is redundant with the others and (6.5.8) so that $l_7, l_8, l_9, l_{10}, l_{11}, l_{12}$ is solution to the invertible system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_7 \\ l_8 \\ l_9 \\ l_{10} \\ l_{11} \\ l_{12} \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{N} - l_1 - l_6 \\ l_6 \\ 1 - \frac{1}{N} - l_2 - l_6 \\ 1 - \frac{2}{N} - l_3 \\ 1 - \frac{1}{N} - l_4 \\ l_1 \end{pmatrix}. \tag{6.5.9}$$

The labels on the inner edges are thus uniquely determined as linear combinations of the labels on the outer edges.

□

We now give a definition of hexagon rotations that incorporates the label map of a quasi hive.

Definition 6.5.22 (Rotation map)**.** Let C, C' be two color maps that differ by a hexagon rotation $h \to h'$ that is, $C'(e) = C(rot(e))$ where $rot : E_{n,d} \to E_{n,d}$ is the permutation of edges induces by the rotation mapping h to h'. Define

$$Rot[C \to C'] : \widetilde{L}^C(\lambda, \mu, \nu, N) \to \widetilde{L}^{C'}(\lambda, \mu, \nu, N) \tag{6.5.10}$$

$$L \mapsto Rot[C \to C'](L) = L' \tag{6.5.11}$$

by setting $L'(e) = L(e)$ for $e \in E_{n,d} \setminus E_h^{\circ}$ and extending L' to $E_h^{\circ} = E_{h'}^{\circ}$ by Lemma 6.5.21.

For notation convenience, we will call a hexagon rotation the image of a map of Definition 6.5.22 for some ABC hexagon h inside a quasi hive.

Lemma 6.5.23 (Rotation is affine bijection)**.** *Let C, C' be two color maps that differ by a hexagon rotation. The map $R[C \to C'] : \widetilde{L}^C(\lambda, \mu, \nu, N) \to \widetilde{L}^{C'}(\lambda, \mu, \nu, N)$ is an affine bijection with coefficients in $\mathbb{Z}[\lambda_i, \mu_i, \nu_i, 1/N]$ and whose inverse is $R[C' \to C]$.*

Proof. For every edge e not interior to h, one has $L'(e) = L(e)$. In particular, $L'(e) = L(e)$ for $e \in \partial h$. Since the matrix of (6.5.9) has integer coefficients and is lower triangular, the values $\{L'(e), e \in E_h^\circ\}$ are integer combinations of the values $\{L(e) = L'(e), e \in \partial h\}$ and $\frac{1}{N}\mathbb{Z}$. $\square$

Figure 6.24 shows an example of a hexagon rotation and the corresponding affine map $L \mapsto L'$.

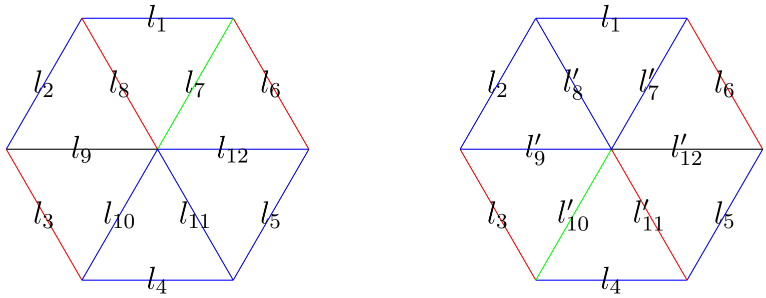

Figure 6.24: Action of a rotation on labels of inner edges.

Using face summation constraints together with equality constraints in the rigid lozenge in h' one has

$$\begin{aligned} l'_9 &= l_4, & l'_7 &= 1 - \frac{1}{N} - l_6 - l'_{12} = \frac{1}{N} + l_3 + l_5 - l_6, \\ l'_{11} &= l_3, & l'_8 &= 1 - \frac{2}{N} - l_1 - l'_7 = \frac{1}{N} + l_3 + l_5 - l_6, \\ l'_{12} &= 1 - \frac{2}{N} - l_3 - l_5, & l'_{10} &= 1 - \frac{1}{N} - l_4 - l'_3. \end{aligned}$$

which is a affine combination of values of L with integer coefficients.

For two color maps C, C' that differ by more than one hexagon rotation, we denote $Rot[C \to C']$ the composition of maps in Definition 6.5.22 for each hexagon rotation needed to go from C to C_0 and then from C_0 to C' where the existence of such paths was given by Proposition 6.5.16. Note that there might be multiple rotation paths from C to C' so that such a map is not unique.

Let $I \subset E_{n,d}$ be the set of edges of $E_{n,d}$ that are not in a rigid lozenge of C_0 except the edges of type 2 between a rigid lozenge and a triangular face with colors $(0, 0, 0)$ and edges of type 1 between a rigid lozenge and a triangular face with colors $(1, 1, 1)$, see Figure 6.25 below for an example. Denote I_0 the edges of I of type 0 where we remove the east-most such edge on each row. By a counting argument, there are $D = \frac{(n-1)(n-2)}{2}$ such edges so that $I_0 = (e_1, \dots, e_D)$.

Lemma 6.5.24 (Simple quasi hive). *For any $z = (z_1, \dots, z_D) \in \left(\frac{1}{N}\mathbb{Z}\right)^D$, there exists a unique label map $\Phi^{C_0}(z) \in L^{C_0}(\lambda, \mu, \nu, N)$ such that for all $1 \leqslant i \leqslant D$: $\Phi^{C_0}(z)(e_i) = z_i$. Moreover, for all $1 \leqslant i \leqslant D$, $\Phi^{C_0}(z)(e_i)$ is given by an affine combination with integer coefficients of $(z_1, \dots, z_D, \lambda, \mu, \nu, \frac{1}{N})$.*

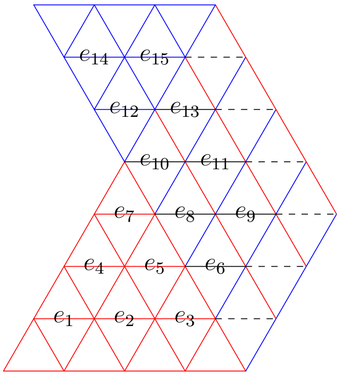

Figure 6.25: The region I for $n = 7$ and $d = 3$. $I_0 = \{e_i, 1 \leqslant i \leqslant 15 = D\}$. Dashed edges are edges of type 0 in $I \setminus I_0$.

Proof. Let $L : \partial E_{n,d} \cup I_0 \to \frac{1}{N}\mathbb{Z}$ be a function satisfying the boundary condition (λ, μ, ν, N) as in (6.5.6). We will show that L can be extended to a quasi label $E_{n,d}$ in a unique way. For any edge of type 2 part of an rigid lozenge, there exists an edge $e_\partial(e) \in \partial E_{n,d}$ of the same type obtained by translation of e by a multiple of $\mathrm{e}^{i\frac{2\pi}{3}}$. Likewise, for any edge of type 1 part of an rigid lozenge, there exists an edge $e_\partial(e) \in \partial E_{n,d}$ of the same type obtained by translation of e by a multiple of $\mathrm{e}^{i\frac{\pi}{3}}$. Assign $L(e) = L(e_\partial(e))$ for each such edge e. By the equality condition on opposite edges of rigid lozenges, any quasi label map has the same values on these edges.

It remains to extend L to edges $e \in I$. The values of ∂I are already determined uniquely by the boundary conditions. Set $L(e_i) = z_i$ for $1 \leqslant i \leqslant D$. We call a band the following configuration of adjacent faces where the west-most triangular face has both its edges of type 0 and 2 labeled, the east-most triangular face has its type 1 edge labeled and all other faces in between have their type 0 edge labeled.

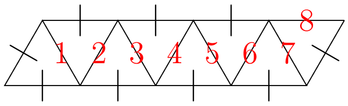

Figure 6.26: A band of size four. Marked edges are the already labeled edges. The labels of other edges are determined in the order of the red numbers by face summation constraints.

We claim that there in a unique labeling of the edges in the band such that the face summation constraints hold. The west-most face of the band has two of its three edges labeled so that the third label is determined uniquely. The south pointing triangular face east to it has two out of three edges labeled so that the third one is also fixed. By inductively propagating east, one labels the edges of the band. Note that the label of the last east-most edge of type 0 is determined. This is why we do not require to fix the values of type 0 edges in $I \setminus I_0$.

The previous paragraph shows that one can extend L to all edges in the south most band of I. Notice then that the region above is also a band. By induction, one extends L to the $n-d$ south most bands of I. The remaining region consists of bands turned upside down which are also uniquely determined by the same reasoning. The resulting map L on $E_{n,d}$ satisfies the face summation constraints so that $L \in \widetilde{L}^{C_0}(\lambda, \mu, \nu, N)$. Since the band completions are unique at each step, the labels of any other quasi label map $L' \in \widetilde{L}^{C_0}(\lambda, \mu, \nu, N)$ with same values on I_0 would agree with L. $\square$

Definition 6.5.25 (Label map associated to edge coordinates)**.** Let us define $\Phi^{C_0}(z) \in L^{C_0}(\lambda, \mu, \nu)$ to be the unique label map constructed in Lemma 6.5.24 from specifying edge coordinates $z \in \left(\frac{1}{N}\mathbb{Z}\right)^D$ on I_0. Let $C : E_{n,d} \to \{0, 1, 3, m\}$ be any color map and let $D = (n-1)(n-2)/2$. Define the following map

$$\begin{aligned} \Phi^C : \qquad \left(\frac{1}{N}\mathbb{Z}\right)^D &\to \widetilde{L}^C(\lambda, \mu, \nu, N) \\ z = (z_1, \dots, z_D) &\mapsto \Phi^C(z) = Rot[C_0 \to C](\Phi^{C_0}(z)). \end{aligned}$$

Proposition 6.5.26 (Quasi hive structure)**.** *The map Φ^C of Definition 6.5.25 is bijective. Moreover, for any $z \in \left(\frac{1}{N}\mathbb{Z}\right)^D$ and edge $e \in E_{n,d}$, $\Phi^C(z)(e)$ is an affine combination of $(z_1, \dots, z_D, \lambda, \mu, \nu, \frac{1}{N})$ with integer coefficients.*

Proof. The map Φ^C is the composition of two bijections : $z \mapsto \Phi^{C_0}(z) \in L^{C_0}(\lambda, \mu, \nu, N)$ and $L \mapsto Rot[C_0 \to C](L)$ which are both affine in $(z_1, \dots, z_D, \lambda, \mu, \nu, \frac{1}{N})$ by Lemma 6.5.23 and Lemma 6.5.24. Its inverse is given by $(\Phi^C)^{-1}(L) = Rot[C_0 \to C](L)_{|I_0}$ where to a label map $L \in L^{C_0}(\lambda, \mu, \nu, N)$, $L_{|I_0} = (L(e_1), \dots, L(e_D))$ are the labels of the edges in I_0. $\square$

So far we have defined the maps $Rot[C \to C']$ and Φ^C from $\widetilde{L}^C(\lambda, \mu, \nu, N) \to \widetilde{L}^{C'}(\lambda, \mu, \nu, N)$ and $\left(\frac{1}{N}\mathbb{Z}\right)^D \to \widetilde{L}^C(\lambda, \mu, \nu, N)$ respectively. We will now extend their definitions to quasi hives.

Definition 6.5.27 (Extension to dual hives)**.** Let C, C' be two color maps. We extend the maps $Rot[C \to C']$ and Φ^C of Definitions 6.5.22 and 6.5.25 to quasi hives by

$$Rot[C \to C'] : \widetilde{H}^C(\lambda, \mu, \nu, N) \to \widetilde{H}^{C'}(\lambda, \mu, \nu, N) \tag{6.5.12}$$
$$H = (C, L) \mapsto Rot[C \to C'](H) = (C', Rot[C \to C'](L)), \tag{6.5.13}$$

and

$$\Phi^C : \left(\frac{1}{N}\mathbb{Z}\right)^D \to \widetilde{H}^C(\lambda, \mu, \nu, N) \tag{6.5.14}$$
$$z = (z_1, \dots, z_D) \mapsto (C, \Phi^C(z)). \tag{6.5.15}$$

6.6 Convergence to a volume of hives

The goal of this section is to show that the volume function $J[\gamma|\alpha, \beta]$ from Theorem 6.2.8 can be expressed as volumes of the polytopes defined in Section 6.1. Section 6.6.1 introduces a limiting object for dual hives and Section 6.6.2 shows the convergence of the volume function to the volumes of limit dual hives. Section 6.6.3 establishes the link between limit dual hives and hives $P^g_{\alpha,\beta,\gamma}$ of Definition 6.1.4. The proofs of Theorem 6.1.5 and Corollary 6.1.6 are then presented in Section 6.6.4.

Since $d\mathbb{P}[\gamma+2r|\alpha+r,\beta+r] = d\mathbb{P}[\gamma|\alpha,\beta]$ for $r>0$, assume in this section without loss of generality that $\alpha,\beta,\gamma\in\mathcal{H}_{reg}$ are such that $\alpha_n,\beta_n>0$ and $\gamma_1<1$. In particular, if λ_N,μ_N,ν_N are such that $\frac{1}{N}\lambda_N\to\alpha$, $\frac{1}{N}\mu_N\to\beta$, $\frac{1}{N}\nu_N\to\gamma$, then $\min((\lambda_N)_n,(\mu_N)_n,N-n-(\nu_N)_1)>d+1$ for N large enough. Hence, we will assume in this section that the hives with boundary (λ_N,μ_N,ν_N) are regular in the sense of (6.5.6), see Figure 6.22.

6.6.1 Limit dual hives

This section introduces limit dual hives which will be linked to the limit of quantum Littlewood-Richardson coefficients of Theorem 6.2.8.

Definition 6.6.1 (Limit dual hive)**.** For $\alpha,\beta,\gamma\in(\mathbb{R}_{\geqslant 0})^3$, the *limit dual hive* $H(\alpha,\beta,\gamma,\infty)$ is the set of pairs (C,L) on the hexagon edges $E_{n,d}$ such that :

1. $C: E_{n,d}\to\{0,1,3,m\}$ is a color map,
2. $L: E_{n,d}\to\mathbb{R}_{\geqslant 0}$ is the label map satisfying
   (a) $L(e_1)+L(e_2)+L(e_3)=1$ for every triangular face of F_k,
   (b) if e,e' are edges of same type on the boundary of a same lozenge f,
      i. $L(e)=L(e')$ if the middle edge of f is colored m,
      ii. $L(e)\geqslant L(e')$ if $h(e)>h(e')$.
   (c) The values of L on $\partial E_{n,d}$ are given by (α,β,γ) so that, sorted in decreasing height of edges, see Figure 6.27 below.

$$\ell^{(0,1)} = (1-\alpha_d,\dots,1-\alpha_1), \qquad \ell^{(2,2)} = (\alpha_{d+1},\dots,\alpha_n)$$
$$\ell^{(2,0)} = (1-\beta_d,\dots,1-\beta_1), \qquad \ell^{(1,1)} = (\beta_n,\dots,\beta_{d+1})$$
$$\ell^{(1,2)} = (\gamma_n,\dots,\gamma_{n-d+1}), \qquad \ell^{(2,2)} = (1-\gamma_{n-d},\dots,1-\gamma_1).$$

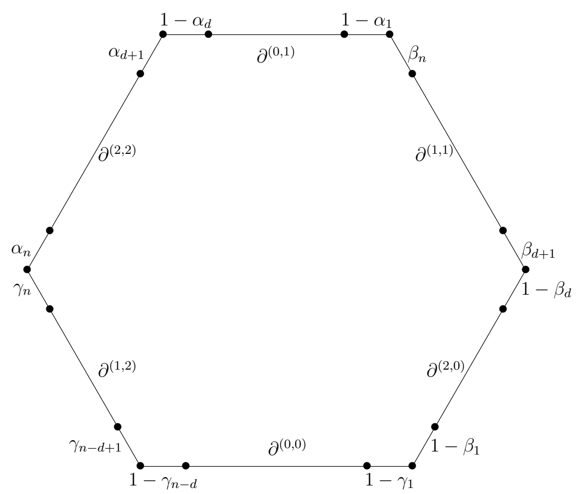

Figure 6.27: Boundary labels for limit hives in $H(\alpha,\beta,\gamma,\infty)$.

As in the discrete case, if C is a color map, we denote by $H^C(\lambda,\mu,\nu,\infty) \subset H(\lambda,\mu,\nu,\infty)$ the subset of limit dual hives of having color map C. As in the previous section, we denote by $\widetilde{H}(\lambda,\mu,\nu,\infty)$ the set of pairs (C,L) as above where we remove the inequality conditions $(2.b.ii)$ on L. Let us denote these inequality constraints by $Ineq(n)$.

Remark 6.6.2 (Maps Φ^C and Rot on limit dual hives)**.** Note that Lemmas 6.5.21, 6.5.23 and 6.5.24 hold for limit quasi dual hives $H = (C,L) \in \widetilde{H}(\lambda,\mu,\nu,\infty)$ extending $z \in \left(\frac{1}{N}\mathbb{Z}\right)^D$ to $z \in \mathbb{R}^D$. Using the same construction as in Section 6.5, we define $Rot[C \to C'] : \widetilde{H}^C(\lambda,\mu,\nu,\infty) \to \widetilde{H}^{C'}(\lambda,\mu,\nu,\infty)$ and $\Phi^C : \mathbb{R}^D \to \widetilde{H}^C(\lambda,\mu,\nu,\infty)$ which are affine bijections with coefficients in $\mathbb{Z}[\alpha,\beta,\gamma]$ obtained by setting $\frac{1}{N} = 0$.

6.6.2 Convergence to a volume

The goal of this part is to prove Proposition 6.6.3 which expresses the limit quantum cohomology coefficients as the volume involving limit dual hives. The proof relies on Lemma 6.6.4 and Lemma 6.6.6 below.

Proposition 6.6.3 (Convergence to volume of dual hives)**.**

$$\lim_{n\to\infty} N^{-D} c_{\lambda_N,\mu_N}^{\nu_N,d} = \sum_C \mathrm{Vol}\left(u \in \mathbb{R}^D, \Phi^C[u] \in H^C(\alpha,\beta,\gamma,\infty)\right). \tag{6.6.1}$$

Recall from Corollary 6.4.4 that

$$N^{-D} c_{\lambda_N,\mu_N}^{\nu_N,d} = N^{-D}|H(\lambda_N,\mu_N,\nu_N,N)| = N^{-D}|\widetilde{H}(\lambda_N,\mu_N,\nu_N,N) \cap Ineq(n)| \tag{6.6.2}$$

$$= \sum_C \int_{\mathbb{R}^D} \sum_{z\in(\frac{1}{N}\mathbb{Z})^D : \Phi^C(z)\in\widetilde{H}(\lambda_N,\mu_N,\nu_N,N)\cap Ineq(n)} \mathbb{1}(u)_{\left\{z+[-\frac{1}{N},\frac{1}{N})^D\right\}} du. \tag{6.6.3}$$

Lemma 6.6.4 (Pointwise convergence)**.** *For any color map C and $N \geqslant 1$, let us define*

$$f_N^C : \mathbb{R}^D \to \mathbb{R}$$
$$u \mapsto \sum_{z\in(\frac{1}{N}\mathbb{Z})^D : \Phi^C(z)\in\widetilde{H}(\lambda_N,\mu_N,\nu_N,N)\cap Ineq(n)} \mathbb{1}(u)_{\left\{z+[-\frac{1}{N},\frac{1}{N})^D\right\}}.$$

Recall that $\frac{1}{N}\lambda_N = \alpha + o(1)$, $\frac{1}{N}\mu_N = \beta + o(1)$ and $\frac{1}{N}\nu_N = \gamma + o(1)$ as $N \to +\infty$. Then, for almost all $u \in \mathbb{R}^D$ with respect to the Lebesgue measure:

$$\lim_{N\to\infty} f_N^C(u) = \mathbb{1}(u)_{\{\Phi^C[u]\in H^C(\alpha,\beta,\gamma,\infty)\}}. \tag{6.6.4}$$

Remark 6.6.5. Note that a priori, $\Phi^C[u] \in \widetilde{H}^C(\alpha,\beta,\gamma,\infty)$ is a label map such that (C,L) is a limit hive of Definition 6.6.1 without the inequality constraints. Here, the right hand side is more restrictive as it requires that $(C,\Phi^C[u]) \in H^C(\alpha,\beta,\gamma,\infty) = \widetilde{H}^C(\alpha,\beta,\gamma,\infty) \cap Ineq(n)$, where $Ineq(n)$ denotes the inequality constraints $(2bii)$ of Definition 6.6.1.

Proof. Take u such that $\Phi^C[u]$ in the interior of $H^C(\alpha,\beta,\gamma,\infty)$. We want to show that $f_N^C(u) = 1$ for $N \geqslant N_0$ which means that one can find a sequence $(z^{(N)}, N \geqslant N_0) = ((z_1^{(N)},\dots,z_D^{(N)}), N \geqslant N_0)$ such that for each $N \geqslant N_0$: $\Phi^C(z^{(N)}) \in \widetilde{H}(\lambda_N,\mu_N,\nu_N,N) \cap Ineq(n)$ and $u \in z^{(N)} + [-1/N, 1/N[$. Let us take the label map

$$L^{(N)} : e \mapsto \Phi_N^C(\lfloor Nu \rfloor/N)(e) \in \frac{1}{N}\mathbb{Z}. \tag{6.6.5}$$

associated to $z^{(N)} = \lfloor Nu \rfloor / N$: $L^{(N)} = \Phi_N^C(z^{(N)})$ where Φ_N^C is associated to the boundaries $(\lambda_N, \mu_N, \nu_N)$. One has $|z^{(N)} - u| < 1/N$ by construction. We need to check that $L^{(N)} \in \widetilde{H}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n)$.

By definition, Φ_N^C is a quasi label map with boundary conditions $(\lambda_N, \mu_N, \nu_N)$ so that $L^{(N)} \in \widetilde{H}(\lambda_N, \mu_N, \nu_N, N)$. Let us check the inequality constraints of $Ineq(n)$. Take any pair of edges (e, e') subject to an inequality. Since $\Phi^C[u] \in H^C(\alpha, \beta, \gamma, \infty)$ is in the interior, this equality is sharp for $\Phi^C[u]$ that is

$$\exists \epsilon_{e,e'} > 0 : \Phi^C[u](e) \leqslant \Phi^C[u](e') + \epsilon_{e,e'}. \tag{6.6.6}$$

Since $\lim_{N\to+\infty} L^{(N)}(e) = \Phi^C[u](e)$ for any edge e, there exists $N_0(e, e')$ such that for $N \geqslant N_0(e, e')$:

$$L^{(N)}(e) \leqslant L^{(N)}(e'). \tag{6.6.7}$$

Hence, $L^{(N)}$ satisfies all the inequality constraints for $N \geqslant N_0$, where N_0 is the largest of the thresholds $N_0(e, e')$ for (e, e') related by an inequality constraint in Definition 6.6.1. Therefore,

$$\forall N \geqslant N_0 : L^{(N)} \in \widetilde{H}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n). \tag{6.6.8}$$

so that $\lim_{N\to+\infty} f_N^C(u) = 1$ as desired. For $\Phi^C[u] \notin H^C(\alpha, \beta, \gamma, \infty)$, one of the inequalities in $(2.b.ii)$ of Definition 6.6.1 is violated, for all other conditions being satisfied by construction of Φ^C. Let (e, e') be a pair of edge such that $(2.b.ii)$ is not satisfied : $\Phi^C[u](e) < \Phi^C[u](e')$ while $h(e) > h(e')$ for some pair of edges of same type adjacent to a same lozenge. Using that $\lim_{N\to+\infty} L^{(N)}(e) = \Phi^C[u](e)$, one has that for N large enough $L^{(N)}(e) < L^{(N)}(e')$ so that $L^{(N)} \notin \widetilde{H}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n)$. Hence, (6.6.4) holds for almost all $u \in \mathbb{R}^D$ with respect to the Lebesgue measure.

□

Lemma 6.6.6 (Uniform bound)**.** *Let $f_N^C(u)$ be as in (6.6.4). Then, there exists a compact $K^{(C)} \subset \mathbb{R}^D$ such that for every $N \geqslant 1$,*

$$|f_N^C(u)| \leqslant \mathbf{1}_{\{u \in K^{(C)}\}}. \tag{6.6.9}$$

Proof of Lemma 6.6.6. If $C = C_0$, the values $z \in \mathbb{Z}^D$ such that

$$\Phi_N^{C_0}(z) \in \widetilde{H}^{C_0}(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n)$$

are in $[0,1]^D$ since $(z_1, \ldots, z_D)$ are the values of $(\Phi_N^{C_0}(z)(e_1), \ldots, \Phi_N^{C_0}(z)(e_D))$ for some horizontal edges $(e_1, \ldots, e_D) \in E_{n,d}$ which are in $[0,1]$ by construction.
If $C \neq C_0$ by definition

$$\Phi^C(z) = Rot[C_0 \to C](\Phi^{C_0}(z)) \tag{6.6.10}$$

where $\Phi^{C_0}(z)$ is the label map of the quasi hive with simple color map C_0 having horizontal edge labeled z. For $z \in \mathbb{Z}^D$ such that $\Phi_N^C(z) \in \widetilde{H}^C(\lambda_N, \mu_N, \nu_N, N) \cap Ineq(n)$, we know from $Ineq(n)$ that values $\{\Phi_N^C(z)(e), e \in E_{n,d}\}$ are in $[0,1]^{E_{n,d}}$. Applying the affine hence continuous map $Rot[C \to C_0]$, we get that $Rot[C \to C_0]\Phi^C(z) = \Phi^{C_0}(z) \in Rot[C \to C_0]([0,1]^{E_{n,d}})$ which is compact. In particular, the labels $(z_i = \Phi^{C_0}(z)(e_i), 1 \leqslant i \leqslant D)$ of horizontal edges e_i in I are in compact sets hence bounded.

□

Proof of Proposition 6.6.3. By Lemma 6.6.4 and Lemma 6.6.6, using dominated convergence theorem in (6.6.3):

$$\lim_{n\to\infty} N^{-D} c^{\nu_N,d}_{\lambda_N,\mu_N} = \sum_C \mathrm{Vol}\left(u \in \mathbb{R}^D, \Phi^C[u] \in H^C(\alpha,\beta,\gamma,\infty)\right). \tag{6.6.11}$$

□

6.6.3 Volume preserving map

This subsection aims at proving that there is a volume preserving map between dual hives $H(\alpha,\beta,\gamma,\infty)$ and hives $P^g_{\alpha,\beta,\gamma}$ of Definition 6.1.4. Refer to the notations of Section 6.1 for the hives and related notions.

Let $H = (C, L) \in \tilde{H}^C(\alpha,\beta,\gamma,\infty)$. We assign to H an function $\Psi^C(H) : R_{d,n} \to \mathbb{R}$ constructed as follows. Set $\Psi^C(H)(v) = d$ where v is the south-east vertex of $R_{d,n}$. For any other vertex $v \in R_{d,n}$ such that $e = (u, v) \in E_{n,d}$ and for which $\Psi^C(H)(u)$ has been set, the value $\Psi^C(H)(v)$ is given by

$$\Psi^C(H)(v) = \begin{cases} \Psi^C(H)(u) + L(e) & \text{if } e \text{ is of type 1 or 2} \\ \Psi^C(H)(u) + 1 - L(e) & \text{if } e \text{ is of type 0.} \end{cases} \tag{6.6.12}$$

See Figure 6.28 for a picture of the recursive construction of Φ^C along edges.

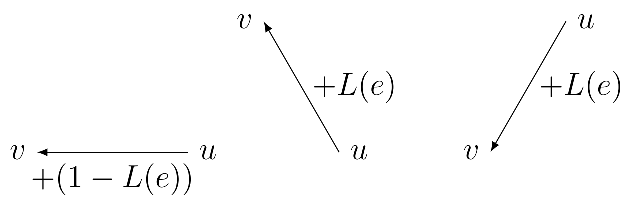

Figure 6.28: Values at vertices when traversing an edge $e = (u, v)$.

Definition 6.6.7 (Dual hive to hive)**.** Let C be a color map. Define

$$S^C = \{v_4 \mid l = (v_1, v_2, v_3, v_4) \text{ is a rigid lozenge}\} \subset R_{d,n} \tag{6.6.13}$$

and

$$\tilde{P}^C_{\alpha,\beta,\gamma} \coloneqq \{f : R_{d,n} \setminus S^C \to \mathbb{R} \mid f_{\partial R_{d,n}} \text{ given by } \alpha,\beta,\gamma\}. \tag{6.6.14}$$

Moreover, define

$$\Psi^C : \tilde{H}^C(\alpha,\beta,\gamma,\infty) \to \tilde{P}^C_{\alpha,\beta,\gamma} \tag{6.6.15}$$
$$H = (C, L) \longmapsto \Psi^C(H) \tag{6.6.16}$$

where $\Psi^C(H)$ is given by (6.6.12) in the above construction.

Remark that the choice of v_4 and the coloring on the boundary ensures that v_4 is never on the boundary of $R_{d,n}$ in the above definition. That the map Ψ^C is well defined is due to the face summation constraint $L(e_1) + L(e_2) = 1 - L(e_0)$ around every face $f \in F_k$ having edges (e_0, e_1, e_2) of respective types $0, 1, 2$ for the label maps L of dual hives in $\tilde{H}^C(\alpha,\beta,\gamma,\infty)$. Remark that Ψ^C depends on C only through its domain and target space.

Remark 6.6.8 (Extension of functions of $\tilde{P}^C_{\alpha,\beta,\gamma}$)**.** Let $f \in \tilde{P}^C_{\alpha,\beta,\gamma}$. Then f can be uniquely extended to a map $f : R_{d,n} \to \mathbb{R}$ by setting $f(v_4) = f(v_3) + f(v_1) - f(v_2)$ for any $v_4 \in S^C$.

Let $C : E_{n,d} \to \{0, 1, 3, m\}$ be a regular color map. Define

$$\begin{aligned} g[C] : R_{d,n} &\to \mathbb{Z}_3 \\ v &\mapsto g[C](v) \end{aligned}$$

where the value at vertex $v \in R_{d,n}$ is set as follows. If $v = A_0$ the south-east most vertex in $R_{d,n}$, set $g[C](v) = 0$. Orient the edges in $E_{n,d}$ around direct triangles clockwise and edges around reversed triangles counterclockwise. For any oriented edge $e = (u, v)$, set

$$g[C](v) = \begin{cases} g[C](u) + 1 & \text{if } C((u, v)) = 1 \\ g[C](u) + 2 & \text{if } C((u, v)) = 0 \\ g[C](u) & \text{if } C((u, v)) \in \{3, m\}. \end{cases} \tag{6.6.17}$$

Proposition 6.6.9 (From color maps to regular labelings)**.** *The above map $C \mapsto g[C]$ is a bijection between color maps on $E_{n,d}$ and regular labelings on $R_{d,n}$. For any regular labeling g, its inverse is given by*

$$\begin{aligned} C[g] : E_{n,d} &\to \{0, 1, 3, m\} \\ e = (u, v) &\mapsto C[g](e) \end{aligned}$$

where, if $w \in R_{d,n}$ denotes the third vertex so that (u, v, w) is a direct triangular face,

$$C[g](e) = \begin{cases} 1 & \textit{if } g(v) - g(u) = 1 \\ 0 & \textit{if } g(v) - g(u) = 2 \\ 3 & \textit{if } g(v) = g(u) \textit{ and } g(w) = g(u) - 1 = g(v) - 1 \\ m & \textit{if } g(v) = g(u) \textit{ and } g(w) = g(u) + 1 = g(v) + 1. \end{cases} \tag{6.6.18}$$

Proof. Let us show that $g[C]$ is well defined. Since C is a color map, the only colors around a triangular face in $E_{n,d}$ are up to cyclic permutations $(0, 0, 0), (1, 1, 1), (1, 0, 3)$ and $(0, 1, m)$. One checks that summing the clockwise differences of values of g going from a vertex to itself around any such color triple gives a zero contribution in $\mathbb{Z}_3$. Therefore, the value of $g[C](v)$ does not depend on the choice of the path from A_0 to v. That $(g[C]^A, g[C]^B, g[C]^C)$ has the right boundary conditions is due to the fact that C is regular. It remains to check the lozenge condition on $g[C]$ from Definition 6.1.1. Take any lozenge $l = (v^1, v^2, v^3, v^4)$ and suppose that $g[C](v^2) = g[C](v^4)$. Note that from Figure 6.3, the edge between v^2 and v^4 is always oriented from v^4 to v^2. The edge $e = (v^4, v^2)$ has color either 3 or m. Since C is a color map, the two faces adjacent to e have either $(1, 0, 3)$ or $(0, 1, m)$ colors. The face with vertices (v^1, v^2, v^4), respectively (v^3, v^2, v^4) is always direct, respectively reversed, see Figure 6.3. If $C(e) = 3$, $g[C](v^1) = g[C](v^2) + 1$ and $g[C](v^3) = g[C](v^2) + 2$ whereas if $C(e) = m$, $g[C](v^1) = g[C](v^2) + 2$ and $g[C](v^3) = g[C](v^2) + 1$. In both cases, $\{g[C](v^1), g[C](v^3)\} = \{g[C](v^2) + 1, g[C](v^2) + 2\}$ and thus $g[C]$ is a regular labeling.

Let us show that $g \mapsto C[g]$ maps a regular labeling g to a color map. Since g is regular, $C[g]$ also is by the same argument as above. Let us show that the only cyclic colors triples around any triangular face are $(0, 0, 0), (1, 1, 1), (1, 0, 3)$ and $(0, 1, m)$. Take any triangular face and denote X, Y, Z the clockwise differences of values of g. Then,

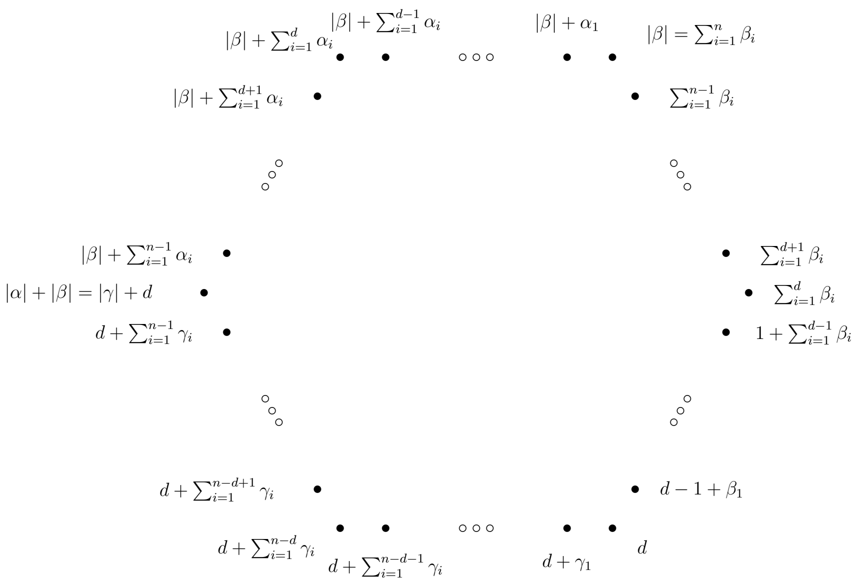

Figure 6.29: Boundary conditions on $P^{g[C]}_{\alpha,\beta,\gamma}$ induced by Ψ^C on boundary conditions in Figure 6.27.

$X + Y + Z = 0[3]$ so that $(X, Y, Z) \in \{(1, 1, 1), (2, 2, 2), (0, 1, 2), (0, 2, 1)\}$ up to cyclic rotation. Note that we exclude $(0, 0, 0)$ by the lozenge condition in Definition 6.1.1 since no lozenge can have three vertices with equal values, for otherwise the second condition of Definition 6.1.1 is violated. These possible height differences give the clockwise colors $\{(1, 1, 1), (0, 0, 0), (3, 1, 0), (m, 0, 1)\}$ respectively. Therefore, $C[g]$ is a color map and by construction $g \mapsto C[g]$ is the inverse of $C \mapsto g[C]$. $\square$

Lemma 6.6.10 (Image of limit dual hives are toric concave functions)**.** *For any regular color map C,*

$$\Psi^C(H^C(\alpha, \beta, \gamma, \infty)) = P^{g[C]}_{\alpha,\beta,\gamma}. \tag{6.6.19}$$

where $g[C]$ is the regular labeling associated to C as in Proposition 6.6.9 and $P^{g[C]}_{\alpha,\beta,\gamma}$ is the polytope defined in Definition 6.1.4.

Proof. The image $\Psi^C(H)$ of any limit hive $H = (C, L) \in H^C(\alpha, \beta, \gamma, \infty)$ can be extended by Remark 6.6.8 to a function $f : R_{d,n} \to \mathbb{R}$ such that by construction

$$\Psi^C(H) = f_{|R_{d,n} \setminus S^C} = f_{|Supp(g[C])} \,.$$

Let us check that $f \in P^{g[C]}_{\alpha,\beta,\gamma}$. By construction, the values of f on $\partial R_{d,n}$ are as in Definition 6.1.4, see Figure 6.29. By definition of the extension, f satisfies the equality constraints over any rigid lozenge in $R_{d,n}$. For any other lozenge $l = (v_1, \dots, v_4)$, the inequality $f(v_2) + f(v_4) \geqslant f(v_1) + f(v_3)$ is equivalent to the inequality (2c) of Definition 6.6.1.

Conversely, to any function $f \in P^{g[C]}_{\alpha,\beta,\gamma}$, associate the label map $L[f] : E_{n,d} \to \mathbb{R}_{\geqslant 0}$, $e = (u, v) \mapsto L[f](e) = f(v) - f(u)$ if e has type 1 or 2 and $L[f](e) = 1 - (f(v) - f(u))$ if e has type 0. The equality and inequality constraints in Definition 6.6.1 are equivalent to the rhombus concavity of f so that $H = (C, L) \in H^C(\alpha, \beta, \gamma, \infty)$. Moreover, we have that $\Psi^C(H) = f$. $\square$

With the definitions above, we have an affine map $\Psi^C \circ \Phi^C : \mathbb{R}^D \to \tilde{P}^C_{\alpha,\beta,\gamma}$. In the rest of this section, we write $\det(\Psi^C \circ \Phi^C)$ for the determinant of the linear part of this application.

Proposition 6.6.11 (Volume preservation by duality)**.** *Let C be a color map. Then, the map*

$$\Psi^C \circ \Phi^C : \mathbb{R}^D \to \tilde{P}^C_{\alpha,\beta,\gamma} \tag{6.6.20}$$

satisfies

$$\left|\det\left(\Psi^C \circ \Phi^C\right)\right| = 1 \tag{6.6.21}$$

and thus

$$\mathrm{Vol}\left(u \mid \Phi^C(u) \in H^C(\lambda,\mu,\nu,\infty)\right) = \mathrm{Vol}\left(u \mid \Psi^C \circ \Phi^C(u) \in P^{g[C]}_{\alpha,\beta,\gamma}\right) = \mathrm{Vol}(P^{g[C]}_{\alpha,\beta,\gamma}). \tag{6.6.22}$$

Proof. For $C = C_0$, enumerat by $e_1, \dots, e_D$ the horizontal edges in C_0 as in Lemma 6.5.24. Then, for $u \in \mathbb{R}^D$ and $v \in R_{d,n}$,

$$[\Psi^{C_0} \circ \Phi^{C_0}(u)](v) = \sum_{e_i}(1-u_i) + (d-v_2)^+ + \sum_{i=1}^{v_2}\beta_i \ ,$$

where the sum is over edges e_i of type 0 connecting v to the east boundary of $E_{n,d}$ with inverse

$$[\Psi^{C_0} \circ \Phi^{C_0}(f)]^{-1}(i) = 1 - (f(v) - f(v'))$$

where v, v' the both endpoint of e_i with the correct orientation. Since $\Psi^{C_0} \circ \Phi^{C_0}$ and $[\Psi^{C_0} \circ \Phi^{C_0}(f)]^{-1}$ have integer coefficients, $\det(\Psi^{C_0} \circ \Phi^{C_0}) = 1$.

If C is general, introduce for each hexagon rotation $C \to C'$ given by an hexagon h the map $\tilde{R}_{C\to C'} : \tilde{P}^C_{\alpha,\beta,\gamma} \to \tilde{P}^{C'}_{\alpha,\beta,\gamma}$ by

1. For $f \in \tilde{P}^c_{\alpha,\beta,\gamma}$, extend f uniquely to a function $f : R_{d,n} \to \mathbb{R}$,
2. Let us describe how the hexagon rotation $C \to C'$ maps f to another function $f' : R_{d,n} \to \mathbb{R}$. The value of the center vertex c of h is uniquely determined by the position of the rigid lozenge in h and the values of f on ∂h. Indeed, if (v, v', v'') are the three other vertices of the rigid lozenge such that $C((c, v'')) = m$, then $f(c) = f(v) + f(v') - f(v'')$. Note that $v, v', v'' \in \partial h$. For every vertex $u \in R_{d,n}$ other that the center vertex c of h, we set $f'(u) = f(u)$. In the hexagon rotation $C \to C'$, the position of the rigid lozenge changes and we set $f'(c) = f(w) + f(w') - f(w'')$ where $w, w', w'' \in \partial h$ are the new vertices of the rigid lozenge in the rotated hexagon.
3. The map $\tilde{R}_{C\to C'}(f)$ is defined as the restriction of f' of the previous step to $R_{d,n} \backslash S^{C'}$.

Note that the map $\tilde{R}_{C\to C'}$ is an affine bijection with integer coefficients whose inverse is given by $\tilde{R}_{C'\to C}$. Let us check that the following diagram is commutative

$$\begin{tikzcd} \tilde{P}^C_{\alpha,\beta,\gamma} \arrow[r, "\tilde{R}_{C\to C'}"] & \tilde{P}^{C'}_{\alpha,\beta,\gamma} \\ \tilde{H}^C(\alpha,\beta,\gamma,\infty) \arrow[u, "\Psi^C"] \arrow[r, "Rot[C,C']"] & \tilde{H}^{C'}(\alpha,\beta,\gamma,\infty) \arrow[u, "\Psi^{C'}"'] \end{tikzcd} \tag{6.6.23}$$

Let $H = (C, L) \in \tilde{H}^C(\alpha, \beta, \gamma, \infty)$ having an ABC hexagon h with center vertex c. Denote by C' the color map obtained after any rotation $h \mapsto h'$ and set $H' = (C', L') = Rot[C, C'](H)$. For any vertex $u \neq c \in R_{d,n}$, one has $\tilde{R}_{C \to C'}(\Psi^C(H))(u) = \Psi^C(H)(u)$. Moreover, if $u \neq c$ then one can find a path of edges from the south-east most vertex of $R_{d,n}$ to u without any edge incident to c. Since the labels of these edges are not changed by $Rot[C \to C']$, we have that $\Psi^{C'}(H')(u) = \Psi^C(H)(u) = \tilde{R}_{C \to C'}(\Psi^C(H))(u)$ as desired. It remains to check that the same property holds for $u = c$. Denote (v, v', v''), respectively (w, w', w'') the vertices on ∂h such that up to cyclic rotation (v, c, v', v''), respectively (w, c, w', w''), are vertices of the rigid lozenge in h, respectively h', and that $C((u, v'')) = C'((u, w'')) = m$. Then,

$$\tilde{R}_{C \to C'}(\Psi^C(H))(u) = \Psi^C(H)(w) + \Psi^C(H)(w') - \Psi^C(H)(w'')$$

Since the values of Ψ do not depend on the chosen path, let us choose the following four paths. Take any path $p = (e_1, \ldots, e_r) \in (E_{n,d})^r$ from the south-east vertex A_0 to w such that for each $1 \leqslant i \leqslant r$, e_i is not an interior edge of h. Then,

1. To evaluate $\Psi^C(H)(w)$, we choose the path p,
2. To evaluate $\Psi^C(H)(w'')$, we append the edge $(w, w'') \in \partial h$ to p,
3. To evaluate $\Psi^C(H)(w')$, we append edges $(w, w''), (w'', w') \in (\partial h)^2$ to p,
4. To evaluate $\Psi^{C'}(H')(u)$, we append the edge (w, u) to p,

which gives

$$\begin{aligned}
\Psi^C(H)(w) &= \sum_{1 \leqslant i \leqslant r} L(e_i) + d \\
\Psi^C(H)(w'') &= \sum_{1 \leqslant i \leqslant r} L(e_i) + L((w, w'')) + d \\
\Psi^C(H)(w') &= \sum_{1 \leqslant i \leqslant r} L(e_i) + L((w, w'')) + L((w'', w')) + d \\
\Psi^{C'}(H')(u) &= \sum_{1 \leqslant i \leqslant r} L'(e_i) + L'((w, u)) + d.
\end{aligned}$$

Since we have chosen edges $e_i \in p$ not interior to h, $L(e_i) = L'(e_i)$ for $1 \leqslant i \leqslant r$. The commutativity of the diagram is thus equivalent to

$$L'((w, u)) = L((w, w'')) + L((w'', w')) - L((w, w'')) = L((w'', w')),$$

i.e

$$L'((w, u)) = L((w'', w')).$$

Notice that $(w, u), (w'', w')$ are two edges of the same type in the rigid lozenge in C' which implies that $L'((w, u)) = L'((w'', w')) = L((w'', w'))$, where the last equality is due to the fact that $(w', w'') \in \partial h$ so that its label value is unchanged by $Rot[C, C']$. The commutativity of (6.6.23) is thus showed.

Using (6.6.23), we have for any sequence of hexagon rotations $C_0 \to C_1 \to \cdots \to C$,

$$\prod \tilde{R}_{C_i \to C_{i+1}} \Psi^{C_0} \circ \Phi^{C_0} = \Psi^C \prod Rot[C_i \to C_{i+1}] \Phi^{C_0} = \Psi^C \circ \Phi^C.$$

On the left hand side, every map is affine with integer coefficient and with inverse having integer coefficients, so the same is true on the right hand-side, and thus

$$\left|\det\left(\Psi^C\circ\Phi^C\right)\right|\ ,$$

which yields (6.6.22) using Lemma 6.6.10 for the first equality.

□

6.6.4 Proof of Theorem 6.1.5 and Corollary 6.1.6

Proof of Theorem 6.1.5. By Theorem 6.2.8 we have

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta]=\frac{sf(n-1)(2\pi)^{(n-1)(n-2)/2}\Delta'(\mathrm{e}^{2i\pi\gamma})}{n!\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})}\lim_{N\to\infty}N^{-(n-1)(n-2)/2}c^{\nu_N,d}_{\lambda_N,\mu_N}. \tag{6.6.24}$$

By Proposition 6.6.3 and Proposition 6.6.11,

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta]=\frac{sf(n-1)(2\pi)^{(n-1)(n-2)/2}\Delta'(\mathrm{e}^{2i\pi\gamma})}{n!\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})}\sum_C\mathrm{Vol}\left(u\mid\Phi^C[u]\in H^C(\alpha,\beta,\gamma,\infty)\right) \tag{6.6.25}$$

$$=\frac{sf(n-1)(2\pi)^{(n-1)(n-2)/2}\Delta'(\mathrm{e}^{2i\pi\gamma})}{n!\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})}\sum_C\mathrm{Vol}(P^{g[C]}_{\alpha,\beta,\gamma}) \tag{6.6.26}$$

$$=\frac{sf(n-1)(2\pi)^{(n-1)(n-2)/2}\Delta'(\mathrm{e}^{2i\pi\gamma})}{n!\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})}\sum_{g:R_{d,n}\to\mathbb{Z}_3\text{ regular}}\mathrm{Vol}_g(P^{g}_{\alpha,\beta,\gamma}). \tag{6.6.27}$$

□

Proof of Corollary 6.1.6. From the expression [Wit91, Eq. (4.116)] proven in [JK98], we have

$$\mathrm{Vol}\left[\mathcal{M}(\Sigma_0^3,\alpha,\beta,\gamma)\right]=\frac{\#Z(\mathrm{SU}(n))\,\mathrm{Vol}(\mathrm{SU}(n))}{\mathrm{Vol}((\mathbb{R}/2\pi\mathbb{Z})^{n-1})^3}\sum_{\lambda\in\mathbb{Z}^n_{\geqslant 0}}\frac{1}{\dim V_\lambda}\chi_\lambda(e^{2i\pi\alpha})\chi_\lambda(e^{2i\pi\beta})\chi_\lambda(e^{2i\pi\gamma}),$$

where $Z(\mathrm{SU}(n))$ is the center of $\mathrm{SU}(n)$. From (6.2.3), we deduce that

$$\mathrm{Vol}\left[\mathcal{M}(\Sigma_0^3,\alpha,\beta,\gamma)\right]=\frac{\#Z(\mathrm{SU}(n))\,\mathrm{Vol}(\mathrm{SU}(n))(2\pi)^{n-1}n!}{\mathrm{Vol}((\mathbb{R}/2\pi\mathbb{Z})^{n-1})^3|\Delta(\mathrm{e}^{2i\pi\gamma})|^2}\mathrm{d}\mathbb{P}[-\gamma|\alpha,\beta].$$

Corollary 6.1.6 is then deduced from Theorem 6.1.5 and the fact that $Z(\mathrm{SU}(n))=2^{(n+1)[2]}$, $\mathrm{Vol}(\mathrm{SU}(n))=\frac{(2\pi)^{n(n+1)/2-1}}{\prod_{k=1}^n k!}$ and $\mathrm{Vol}((\mathbb{R}/2\pi\mathbb{Z})^{n-1})=(2\pi)^{n-1}$. □

Chapter 7

Enumeration of crossings in two-step puzzles

In his work [Knu99], Knutson conjectured that the structure constants of the cohomology ring of a partial flag variety $\mathrm{GL}(n)/P$ can be computed by the number of tilings of the triangular lattice called puzzles using specific tiles with side labels. The puzzle conjecture for the two-step flag variety was eventually proved in [Buc+16] as presented in Section 3.2.4. The extension of the puzzle rule to equivariant Schubert structure constants for the two-step flag variety was conjectured by Coskun and Vakil [CV09] and proved by Buch [Buc15]. In the latter, Buch introduced transformations on equivariant two-step puzzles called mutations which in particular encompass the local rules of [Buc+16].

Edge labels on boundaries of two-step puzzles [Buc+16] are 012 strings, see Section 3.2.4. At the scale of the whole puzzle, the labels 0 and 1 create lines starting from the boundaries and crossing each other inside the puzzle. There are two possible types of crossings up to rotations. In one type of crossing, the lines joining identical labels from both sides cross each other by keeping their direction constant which is encoded in the puzzle by the label 7 inside the configuration. In the other type of crossing, the line joining labels 0 may not keep its direction constant which is encoded by the presence of at least one label 3 in the configuration. In the work [FT24] which is Chapter 6 of this thesis, inspired from the hive model of Knutson and Tao [KT99; KTW04], we constructed a bijection between two-step puzzles and objects called *two-colored dual hives* which involve tilings of the triangular lattice called color maps together with edge labels satisfying inequality and equality conditions. The bijection converts labels 7 in two-step puzzles to edges of color m in color maps. Moreover, the number of crossings of the second type is equal to the number of edges of color 3 in the color map.

This chapter presents our result which is a formula for the number of crossings of each type that is, for both the number of labels 7 and the number of crossings of the second type in any two-step puzzle. Our formula depends only on the 012 boundary strings of the puzzle. In Section 7.1 we give the necessary definitions to state the main result. The latter is first expressed in terms of color maps in Theorem 7.1.5 which translates to crossings in two-step puzzles in Corollary 7.1.6. Section 7.2 recalls some definitions of local configurations in color maps. Section 7.3 proves the main identity in a special case where the boundaries of the color map are in a simple form. Section 7.4 starts with local propagations of configurations in color maps called gashes which are directly inspired from [Buc15] and proves the main identity by induction using propagations.

7.1 A formula for the number of crossings

Definition 7.1.1 (Triangular lattice)**.** Let $n \geqslant 1$ and let $\xi = \mathrm{e}^{\frac{i\pi}{3}}$. Let us denote by $T_n = \{r + s\xi, 0 \leqslant r + s \leqslant n\}$ the vertices of the triangular lattice of size n and by $E_n = \{(x, x+v) \mid x, x+v \in T_n \text{ and } v \in \{-\xi^{2l}, 0 \leqslant l \leqslant 2\}\}$ the set of edges in T_n. The faces of the lattice T_n are triangles which are called direct (respectively reversed) if the corresponding vertices $(x_1, x_2, x_3) \in T_n^3$ can be labeled in such a way that $x_2 - x_1 = (1, 0)$ and $x_3 - x_1 = \xi$ (respectively $x_3 - x_1 = \overline{\xi}$).

Edges in E_n can only have three possible orientations. If $x = r + s\xi \in T_n$, we define three coordinates (x_0, x_1, x_2) by

$$x_0 \coloneqq n - (r+s),\ x_1 \coloneqq r \text{ and } x_2 \coloneqq s.$$

Definition 7.1.2 (Edge coordinate and type)**.** We say that an edge $e = (x, x+v)$ *is of type* l for $l \in \{0, 1, 2\}$ when $v = -\xi^{2l}$. The *origin* of e is x and the coordinates of e is the triple $(e_0, e_1, e_2) = (x_0, x_1, x_2)$. The *height* of e of type l is $h(e) = e_l$. Define also the boundary edges of E_n by

$$\begin{aligned}
\partial_0^{(n)} &\coloneqq (((n-r+1, 0), (n-r, 0)), 1 \leqslant r \leqslant n) \\
\partial_1^{(n)} &\coloneqq \Big((n\xi + (r-1)\overline{\xi}, (n\xi + r\overline{\xi})), 1 \leqslant r \leqslant n\Big) \\
\partial_2^{(n)} &\coloneqq \Big(((r-1)\overline{\xi}, r\overline{\xi}), 1 \leqslant r \leqslant n\Big).
\end{aligned}$$

Definition 7.1.3 (Color map)**.** Let $n \geqslant 1$. A *color map* is a map $C : E_n \to \{0, 1, 3, m\}$ such that the boundary colors around each triangular face in the clockwise order is either $(0,0,0)$, $(1,1,1)$, $(1,0,3)$ or $(0,1,m)$ up to a cyclic rotation.

The values of a color map C on the boundary edges are denoted $\partial C = (\partial_0 C, \partial_1 C, \partial_2 C)$ and are defined for $l \in \{0,1,2\}$ as $\partial_l C = C_{|\partial_l^{(n)}}$. We say that C has boundary condition $\partial = (\partial_0, \partial_1, \partial_2)$ if $\partial C = \partial$.
Alternatively, one can view a color map C as a tiling of T_n by the set of edge labeled tiles of Figure 7.1 where tiles can be rotated. The last two tiles are respectively called 3 and m lozenges in accordance with the color of their middle edge.

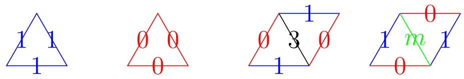

Figure 7.1: Possible tiles for color maps

As there is an equal number of both 0 and 1 labels on each side of two-step puzzles, we will consider boundary conditions $\partial C \in \{0,1\}^{3n}$ having an equal number of 0 and 1 colored edges respectively denoted by n_0 and n_1 so that $n_0 + n_1 = n$, see Figure 7.2 below. Such boundary conditions correspond to those of two-step puzzles [Buc+16] where one removed the labels 2 from the boundary 012 strings.

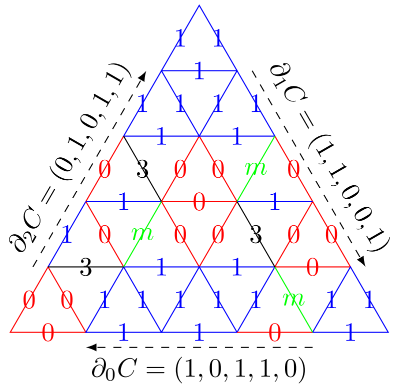

Figure 7.2: A color map on E_5 with boundary condition $\partial C = ((1,0,1,1,0),(1,1,0,0,1),(0,1,0,1,1))$.

Definition 7.1.4 (Gash numbers)**.** Let $C : E_n \to \{0,1,3,m\}$ be a color map. For any $l \in \{0,1,2\}$ and edge $e \in \partial_l^{(n)}$ denote by $n(C,e) = |\{e' \in \partial_l^{(n)} : h(e') < h(e) \text{ and } C(e') = 1\}|$ the number of 1 colored edges east (respectively north, south) to e if $e \in \partial_0^{(n)}$ (respectively $e \in \partial_1^{(n)}$, $e \in \partial_2^{(n)}$). The *gash numbers* of the color map C are defined for $l \in \{0,1,2\}$ as

$$G(C,l) \coloneqq \sum_{e \in \partial_l^{(n)} : C(e)=0} n(C,e). \tag{7.1.1}$$

For instance, in the color map C of Figure 7.2, one has $G(C,0) = 4$, $G(C,1) = 4$, $G(C,2) = 1$. The main result of this chapter is Theorem 7.1.5 which gives a formula for the number of 3 and m colored edges in color maps which depends only on the gash numbers.

Theorem 7.1.5 (Label count in color maps)**.** *Let C be a color map on E_n having n_0, respectively n_1, edges of color 0, respectively 1, on each of its boundaries. Let $m(C)$ and $s(C)$ denote respectively the number of m and 3 colored edges in C. Then,*

$$m(C) = G(C,0) + G(C,1) + G(C,2) - n_0 n_1 \tag{7.1.2}$$

and

$$s(C) = 2n_0 n_1 - G(C,0) - G(C,1) - G(C,2). \tag{7.1.3}$$

Let us also mention that one can count other types of tiles in C. The enumeration of triangular faces having edges of the same color is given in Corollary 7.4.6.

In [FT24, Theorem 5.3] a bijection was defined between two-step puzzles of [Buc+16] and objects called two colored dual hives consisting of a color map together with a label map, see [FT24, Definition 5.1] for details on the definition. In particular, this bijection converts labels 7 of two-step puzzles into to m colored edges. Moreover the number of edges $e \in E_n$ with color $C(e) = 3$ in the obtained color map is equal to the number of pieces of the form of Figure 7.3 where the number of labels 2 is arbitrary and where the configuration can be rotated. This piece is one of the composed puzzle pieces of [Buc+16] which we call a *soft crossing* in the rest of this chapter.
Recall that the clockwise labels on boundaries of two-step puzzles are 012 strings. Let u be a 012 string of length n: $u = u_1 \dots u_n$ where $u_i \in \{0,1,2\}, 1 \leqslant i \leqslant n$. In accordance with Definition 7.1.4, define

$$G(u) \coloneqq \sum_{1 \leqslant i \leqslant n : u_i = 0} |\{j \leqslant i \mid u_j = 1\}|.$$

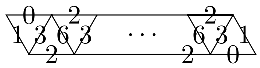

Figure 7.3: A soft crossing in two step puzzles.

Theorem 7.1.5 yields a direct computation of the number of labels 7 and soft crossings in any two-step puzzle given in Corollary 7.1.6.

Corollary 7.1.6 (Labels 7 and soft crossings in two-step puzzles)**.** *Let P be a two-step puzzle with boundary given by three* 012 *strings u, v, w respectively on the left, right and bottom sides in clockwise order, each having n_0 symbol* 0 *and n_1 symbol* 1*. Let $n(P, \mathrm{sc})$ and $n(P, 7)$ denote respectively the number of soft crossings and labels* 7 *in P. Then,*

$$n(P,7) = G(u) + G(v) + G(w) - n_0 n_1,$$
$$n(P,\mathrm{sc}) = 2n_0 n_1 - G(u) - G(v) - G(w).$$

Proof. Let C be the color map associated to the image of P by the bijection from Definition 5.2 in [FT24]. Then, $G(C,0) = G(w)$, $G(C,1) = G(v)$ and $G(C,2) = G(u)$. Moreover, $n(P,\mathrm{sc}) = s(C)$ and $n(7,P) = m(C)$ from which one derives the result using (7.1.2) and (7.1.3). □

Sketch of the proof of Theorem 7.1.5. In Section 7.2, we introduce some transformations on color maps that will play a role in the rest of the chapter. In Section 7.3, we prove Theorem 7.1.5 in the case where $G(C,2) = 0$. This is done by showing that when $G(C,2) = 0$, the color map can be reduced to a simple color map in which the counting is explicit. In Section 7.4, we give a procedure to transform any color map C to another color map C' such that $G(C',2) = G(C,2) - 1$ from which one can prove Theorem 7.1.5 by induction.

7.2 Arrows

In this section, we recall some definitions on local configurations introduced in [FT24].

Definition 7.2.1 (Opening)**.** Let $x \in T_n$. An *opening* of type $l \in \{0,1,2\}$ at x is a pair of edges $(e,e') \in E_n^2$ such that if $e = (e_1,e_2)$, $e' = (e'_1, e'_2)$ with $(e_1, e'_1, e_2, e'_2) \in T_n^4$ and $t(e), t(e')$ are the types of e and e',

$$e_i = e'_i = x \text{ for some } i \in \{1,2\},$$
$$\{t(e), t(e')\} = \{l-1, l+1\} \text{ and } C(e) = C(e') \in \{0,1\}.$$

The *color* of the opening is defined as the color of edges e and e'.

Consider an opening $a = (e,e')$ at x of type l and color $c \in \{0,1\}$. Let $e'' = e''(a)$ be the edge such that e, e' are edges of the lozenge with middle edge e''. The only possible colors of the edge e'' are $C(e'') \in \{0,1\}$. If $C(e'') = c$, the two triangular faces of the lozenge with middle edge e'' have all of their edges colored c. If $C(e'') \neq c$, then there is an opening a' of type l and color c at the other endpoint of e''. Note that there can only be finitely many such openings before $C(e'') = c$.

Definition 7.2.2 (Arrow)**.** Let $a = (e, e')$ be an opening of type l and color c. Let $r \geqslant 0$ be the number of successive openings having middle edge e'' such that $C(e'') \neq c$ with $C(e'') \in \{0, 1\}$ as in the previous paragraph. An *arrow* of length $r \geqslant 0$ at the opening a is the configuration of edges consisting of the $r \geqslant 0$ successive pairs of 3 and m lozenges together with the pair of direct and reverse faces with boundary edges of color c.

See Figure 7.4 for examples of openings and arrows.

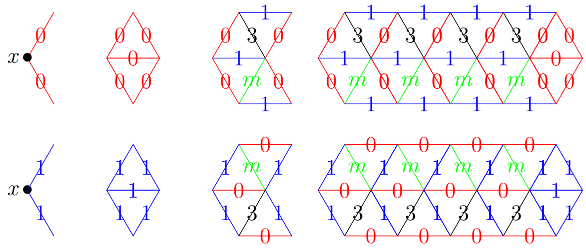

Figure 7.4: First row from left to right : an opening a with color 0 and type 0 at x, the case $C(e'') = c$, the case $C(e'') \neq c$ and an arrow of length $r = 4$. The second row is the analog for color 1.

Let A be an arrow of length $r \geqslant 1$ at an opening with center x. The reversal of A is the configuration obtained by applying a rotation of π to A. An example of arrow reversal is given in Figure 7.5.

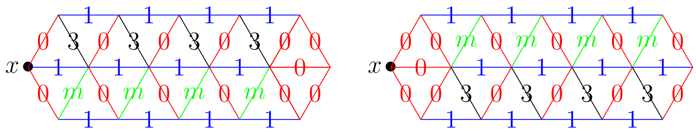

Figure 7.5: Reversal of an arrow of length 4 at x.

7.3 A simpler case

In this section, we prove (7.1.2) for color maps C such that $G(C, 2) = 0$. We first reduce the color map C to C' so that all the 0 colored edges on $\partial_0 C'$ are consecutive starting from the bottom left coner of T_n. This is done in Section 7.3.1. In Section 7.3.2, we give an explicit counting of $m(C')$ and in Section 7.3.3 we show (7.1.2) when $G(C, 2) = 0$ using the two previous sections.

7.3.1 Reduction of color maps

Definition 7.3.1 (Lozenge and trapeze regions)**.** Let $x = (x_0, x_1, x_2) \in T_n$ and let $r, s \geqslant 0$ be such that $(r, s) \neq (0, 0)$. We define the *lozenge region* $L[r, s, x] \subset E_n$ as

$$L[r, s, x] \coloneqq E(\{x + u + v\xi, (u, v) \in \{0, \dots, r\} \times \{0, \dots, s\}\}) \tag{7.3.1}$$

where for a subset $V \subset T_n$, $E(V) \subset E_n$ is the subset of edges having both endpoints in V. Moreover, the *trapeze region* $T[r, s, x] \subset E_n$ is defined for $s \geqslant r$ as

$$T[r, s, x] \coloneqq E(\{x + u + v\xi, (u, v) \in \{0, \dots, r\} \times \{0, \dots, s\} : v + u \leqslant s\}). \tag{7.3.2}$$

For an illustration of lozenge and trapeze regions, see Figure 7.6.

Lemma 7.3.2 (Filling a lozenge region)**.** *Let $L[r,s,x]$ be a region as in (7.3.1). Suppose that its boundary edges $\{(x+u, x+u-1), 1 \leqslant u \leqslant r\}$ and $\{(x+v\xi, x+(v+1)\xi), 0 \leqslant v \leqslant s-1\}$ are colored 1 and 0 respectively. Then, every edge in $L[r,s,x]$ of type 1 has color 3, which determines the color of all edges in $L[r,s,x]$ uniquely.*

Proof. For any $v \in T_n$ such that $C((v+1,v)) = 1$ and $C((v, v+\xi)) = 0$, there is only one possible set of values for a color map C on edges $(v+\xi, v+1), (v+1+\xi, v+\xi), (v+1, v+1+\xi)$ which is given by $(3,1,0)$. Applying this constraint to $v = x, x+1, \ldots, x+r-1$ in this order and using induction to fill the remaining region $L[r, s-1, x+\xi]$ shows the result. ☐

Lemma 7.3.3 (Filling a trapeze region)**.** *Let $T[r,s,x]$ be a region as in (7.3.2). Suppose that its edges $\{(x+u, x+u-1), 1 \leqslant u \leqslant r\}$ and $\{(x+v\xi, x+(v+1)\xi), 0 \leqslant v \leqslant s-1\}$ are colored 0. Then, up to some arrow reversals, every edge in $T[r,s,x]$ has color 0.*

Proof. We will prove the result by induction over r. Assume that $r = 1$. The triangular face having edges $(x+1, x)$ and $(x, x+\xi)$ colored 0 has its third edge $(x+\xi, x+1)$ of type 1 also colored 0. Then, the edges $(x+\xi, x+1)$ and $(x+\xi, x+2\xi)$ form a 0 opening that we call o_1. Consider the arrow A_1 at o_1 of length $\ell \geqslant 0$ having its other endpoint at $x+\ell+1$. Apply the arrow reversal as in Figure 7.5 which only changes the colors of edges inside A_1. Then, the edges $(x+1, x+1+\xi), (x+1+\xi, x+\xi)$ and $(x+2\xi, x+1+\xi)$ have color 0. Notice that in the resulting configuration, the edges $(x+2\xi, x+1+\xi)$ and $(x+2\xi, x+3\xi)$ form a 0 opening. Moreover, reversing an arrow between endpoints x and $x+\ell+1$ does not modify the colors of the edges e having origin y such that $y_0 \geqslant x_0$. By successively considering the 0 openings formed by edges $(x+v\xi, x+1+(v-1)\xi)$ and $(x+v\xi, x+(v+1)\xi)$ for $1 \leqslant v \leqslant s-1$, we get that $T[1,s,x]$ has every edge colored 0.
For $r \geqslant 2$, using the same argument as above shows that all edges in $T[1,s,x]$ are colored 0. Since $T[r,s,x] = T[1,s,x] \cup T[r-1, s-1, x+1]$, one gets the result by induction. ☐

Remark 7.3.4 (Filling lozenge and trapeze regions)**.** Note that the results of Lemmas 7.3.2 and 7.3.3 remain valid if one swaps labels 0 and 1, replacing 3 lozenges in a lozenge region by m lozenges and 0 colored edges in the trapeze region by 1 colored edges.

The next Lemma shows that the bottom region adjacent to $\partial_0^{(n)}$ of a color map has an explicit description in terms of lozenge and trapeze regions. An illustration of that region is given in Figure 7.6 where the lozenge regions are filled with lozenges having middle edge of type 1 colored 3 and trapeze regions have all of their edges colored 0.

Lemma 7.3.5 (Structure above $\partial_0^{(n)}$)**.** *Let C be a color map such that $G(C,2) = 0$. Then, there exists $p = p(C) \geqslant 1$ and $0 = y_0 \leqslant x_1 < y_1 < x_2 < \cdots < y_{p-1} < x_p < y_p \leqslant n$ such that up to some arrow reversals, denoting $r_i = y_i - x_i$ and $b_i = x_i - y_{i-1}$ for $1 \leqslant i \leqslant p$, the regions*

$$L[b_1, n_0, (y_0, 0)], L[b_2, n_0 - r_1, (y_1, 0)], \ldots, L[b_p, r_p, (y_p, 0)]$$

are filled by lozenges having middle edge of type 1 colored 3 and such that regions

$$T[r_1, n_0, (x_1, 0)], T[r_2, n_0 - r_1, (x_2, 0)], \ldots, T[r_p, r_p, (x_p, 0)]$$

have their edges colored 0.

Proof. Set $y_0 = 0$ and by convention $C((0,-1)) = C((n+1, n)) = 1$. Define

$$p = |\{1 \leqslant x \leqslant n-1 \mid C((x, x-1)) = 0 \text{ and } C((x+1, x)) = 1\}| \geqslant 1 \tag{7.3.3}$$

and for $1 \leqslant i \leqslant p$,

$$x_i = \inf\{u \geqslant y_{i-1} \mid C((u, u-1)) = 1 \text{ and } C((u+1, u)) = 0\} \tag{7.3.4}$$
$$y_i = \inf\{u \geqslant x_i \mid C((u, u-1)) = 0 \text{ and } C((u+1, u)) = 1\}. \tag{7.3.5}$$

We have that $y_0 \leqslant x_1 < y_1 < x_2 < \cdots < y_{p-1} < x_p < y_p$. Recall that $r_i = y_i - x_i$ and $b_i = x_i - y_{i-1}$ for $1 \leqslant i \leqslant p$.

Since $G(C, 2) = 0$, the region $L[b_1, n_0, (0, 0)]$ has its edges $\{(x, x-1), 1 \leqslant x \leqslant b_1\}$ and $\{(x, x+\xi), 0 \leqslant x \leqslant n_0 - 1\}$ colored 1 and 0 respectively which implies by Lemma 7.3.2 that it is filled by lozenges having middle edge of type 1 colored 3 except in the case where $b_1 = 0$ for which $L(b_1, n_0, (0, 0)) = \{(x, x+\xi), 0 \leqslant x \leqslant n_0 - 1\}$ has all of its edges on $\partial_2^{(n)}$ colored 0. Remark that edges in $L(b_1, n_0, (0, 0))$ with coordinate e_1 equal to x_1 are colored 0. Therefore, the trapeze region $T[r_1, n_0, (x_1, 0)]$ has its boundary edges colored 0 as in Lemma 7.3.3 which shows that up to arrow reversals, it has all of its edges colored 0. Using Lemmas 7.3.3 and 7.3.2 successively on the regions

$$L[b_1, n_0, (y_0, 0)], L[b_2, n_0 - r_1, (y_1, 0)] \ldots, L[b_p, r_p, (y_{p-1}, 0)]$$

and

$$T[r_1, n_0, (x_1, 0)], T[r_2, n_0 - r_1, (x_2, 0)] \ldots, T[r_p, r_p, (x_p, 0)]$$

gives the result. Notice that the order of the applications of Lemma 7.3.3 is compatible with the arrow reversals involved for the trapeze regions in the sense that arrow reversals in $T[a, b, x]$ only affect edges $e \in E_n$ such that $e_1 \geqslant x_1$.

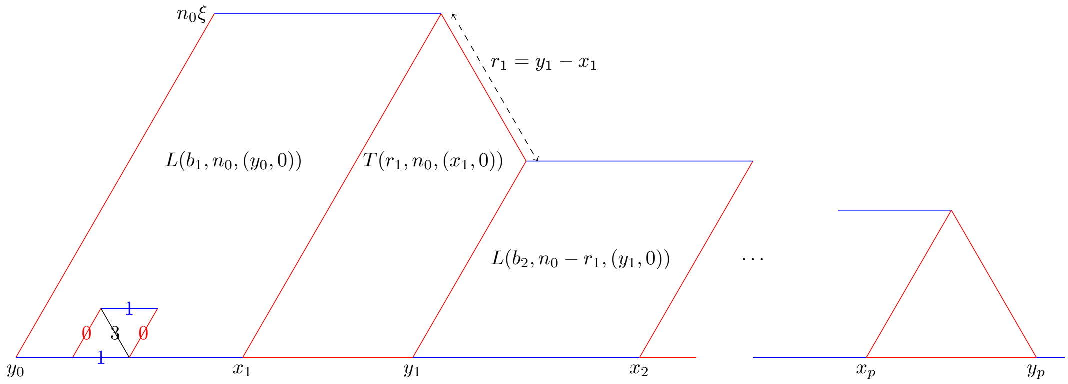

Figure 7.6: Region at the bottom of C. Lozenge regions are filled with 3 lozenges and trapeze regions are filled with 0 colored edges.

□

Recall that for $1 \leqslant i \leqslant p, r_i(C) = y_i - x_i$ denotes the number of edges of $\partial_0 C$ which are in the i-th trapeze region of Figure 7.6.

Lemma 7.3.6 (Grouping columns). *Let C be a color map and let $p = p(C)$ and $0 = y_0 \leqslant x_1 < y_1 < x_2 < \cdots < y_{p-1} < x_p < y_p \leqslant n$ be defined as in Lemma 7.3.5. Assume that $p(C) \geqslant 2$. Using arrow reversal, adding $r_p \times b_p$ m-colored edges and removing $r_p \times b_p$ 3-colored edges, one can map C to C' such that $p' = p(C') = p(C) - 1$ and $r_{p'}(C') = r_p(C) + r_{p-1}(C)$.*

Proof. Let us first define a local transformation. Consider any vertex $v \in T_n$ such that $C((v+\xi-1,v)) = 3$ and $C((v+1,v)) = 0$, so that $C((v+\xi,v+1)) = 0$. Consider the color map C_v where $C_v((v,v-1)) = 0$ and $C_v((v,v+\xi)) = m$. We call $C \mapsto C_v$ the replacement at v, see Figure 7.7. The color map C_v has one less 3 colored edge and one more m colored edge than C.

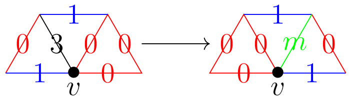

Figure 7.7: Replacement at vertex v.

Consider the lozenge region $L[b_p+2, r_p, (y_{p-1}-1, 0)]$ where $b_p = x_p - y_{p-1}$ and $r_p = y_p - x_p \geqslant 1$, see Figure 7.8. Apply replacements successively at $x_p, \ldots, y_{p-1}+1$ and call C_1 the resulting color map, see Figure 7.9 and Figure 7.10 for an illustration of this step. Notice that C_1 has an arrow of length b_p at the 0 opening at $v_1 = (y_{p-1}, 0) + \xi$. Reverting this arrow creates an arrow at $v_2 = v_1 + \xi$ of length b_p, see Figure 7.11. By reverting arrows with openings at $(y_{p-1}, 0) + q\xi, 1 \leqslant q \leqslant r_p$ each with length b_p, the resulting color map $\varphi(C)$ satisfies

$$p(\varphi(C)) = p \text{ and } r_p(\varphi(C)) = r_p - 1 \text{ if } r_p \geqslant 2, \tag{7.3.6}$$

$$p(\varphi(C)) = p-1 \text{ and } r_{p(\varphi(C))}(\varphi(C)) = r_{p-1} + 1 \text{ if } r_p = 1, \tag{7.3.7}$$

see Figure 7.12. By applying the previous transformation $C \mapsto \varphi(C)$ a number of times equal to r_p, one gets a color map C' such that $p(C') = p-1$ and $r_{p(C')}(C') = r_{p-1} + r_p$.

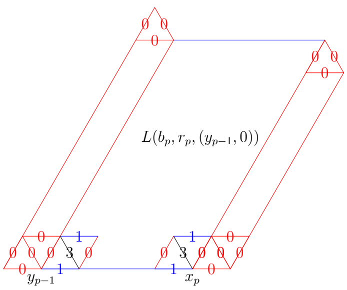

Figure 7.8: Color map C.

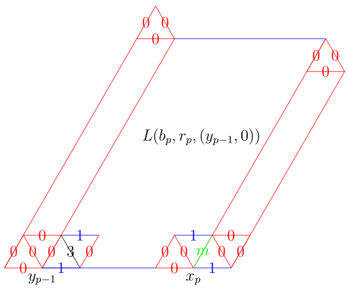

Figure 7.9: Replacement x_p.

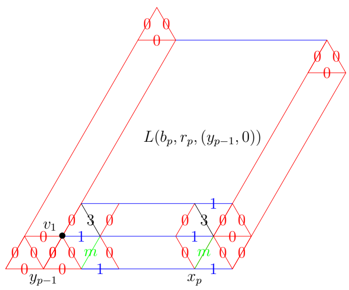

Figure 7.10: Replacements at $x_p, \ldots, y_{p-1}+1$.

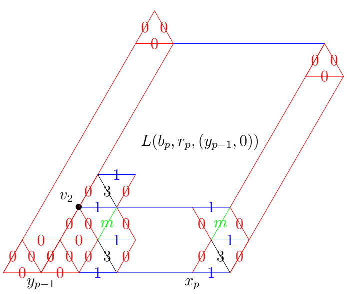

Figure 7.11: First arrow reversal creating an arrow at v_2.

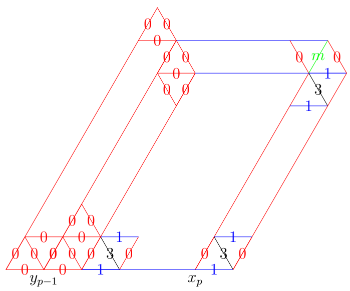

Figure 7.12: Final configuration $\varphi(C)$.

□

From the previous Lemmas, one derives the following result.

Proposition 7.3.7 (Edge count during reduction)**.** *Let C be a color map such that $G(C,2)=0$ and let b_1 be defined as in Lemma 7.3.5. One can reduce C to a color map C' such that $p(C')=1$ and $b_1(C')=0$ by removing (respectively adding) $M=n_0n_1-G(C,0)$ edges of color 3 (respectively of color m) so that $m(C')=m(C)+M$ and $s(C')=s(C)-M$.*

Proof. Apply the transformation of Lemma 7.3.6 until $p=1$. Let us compute the total number M of exchanged 3 and m colored edges in the process. The only tranformation that changes the number of edges of color 3 and m is the replacement as in Figure 7.7. By Lemma 7.3.6, one has applied

$$\tilde{M}=r_pb_p+b_{p-1}(r_{p-1}+r_p)+\cdots+b_2(r_2+\cdots+r_p)=\sum_{j=2}^{p}r_j\sum_{i=1}^{j}b_i-b_1\sum_{j=2}^{p}r_j$$

replacements of 3-colored edges by the same number of m-colored edges. Moreover,

$$G(C,0)=\sum_{j=1}^{p}r_j(n_1-\sum_{i=1}^{j}b_i)=(n_1-b_1)\sum_{j=1}^{p}r_j-\tilde{M}=(n_1-b_1)n_0-\tilde{M}.$$

If the resulting color map $\tilde{C}$ has $b_1(\tilde{C})\geqslant 1$, apply Lemma 7.3.6 a number of times equal to n_0 so that the obtained color map C' has $x_1(\tilde{C})=0$. This last step removed $b_1(C)n_0=b_1(\tilde{C})n_0$ edges of color 3 from $\tilde{C}$ and added the same number of edges of color m. Therefore, one has removed $M=\tilde{M}+b_1n_0=n_0n_1-G(C,0)$ edges of color 3 and added the same number of edges of color m. □

Definition 7.3.8 (Reduced color map)**.** A color map $C:E_n\to\{0,1,3,m\}$ such that $G(C,2)=0$, $p(C)=1$ and $b_1(C)=0$ is called a *reduced color map.*

7.3.2 Structure of reduced color maps

In this section, we only consider reduced color maps as any color map C such that $G(C,2)=0$ can be reduced thanks to Proposition 7.3.7 above. We first show that most of the edges in reduced color map have their color fixed except in some region, see Figure 7.13. This region consists of specific configuration of edges described in Remark 7.3.10. From this we derive the main result of this section in Proposition 7.3.11 which gives the number of m and 3 colored edges in reduced color map.

By Lemma 7.3.3, a reduced color map has every edge of its trapeze region $T[n_0,n_0,(0,0)]$ colored 0. Let $R=(E_n\setminus T[n_0,n_0,(0,0)])\cup\{(n_0+s\xi^2,n_0+(s+1)\xi^2),0\leqslant s\leqslant n_0-1\}$ be the remaining region pictured in Figure 7.13.

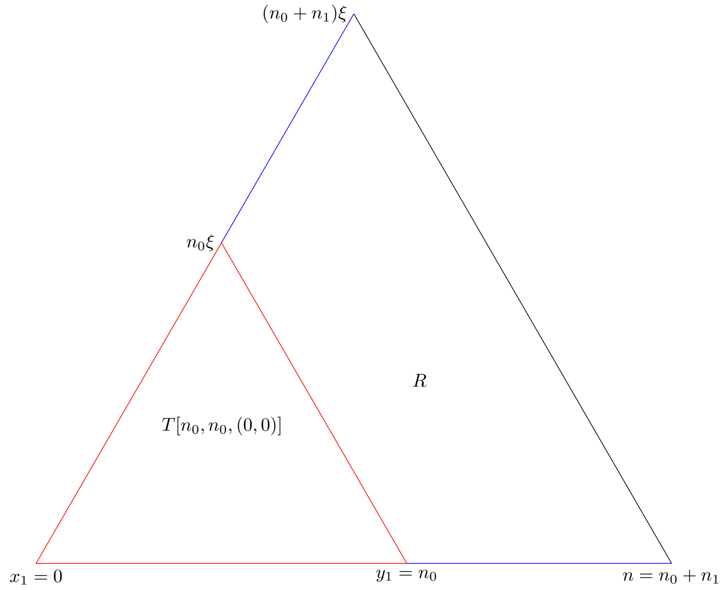

Figure 7.13: The region R in a reduced color map. Edges outside R have their color fixed.

The next Lemma shows that edges of color 0 in R can only be of type 1.

Lemma 7.3.9 (Edges of color 0 in R)**.** *Let R be the region above associated a to reduced color map C. Then, every edge of color* 0 *in R is of type* 1*.*

Proof. Assume for the sake of contradiction that there exists an edge $e^{(0)}$ of type 0 or 2 in R. Take $e^{(0)}$ such that its origin $x^{(0)}$ has minimal coordinate x_2. Since the color map is reduced, edges in $R \cap \partial_0^{(n)}$ have color 1 so that $x_2 \geqslant 1$ for edges of type 0. If $e^{(0)}$ is of type 0, then one of the edges of type 0 with origins $x^{(0)} - \xi, x^{(0)} - \xi + 1$ or the edge of type 2 with origin $x^{(0)} - \xi$ is colored 0. In either case, the minimality of the x_2 coordinate of $x^{(0)}$ is violated. If $e^{(0)}$ is of type 2, the type 2 edge with center $x^{(0)} + 1$ is colored 0 and both are opposite edges of a lozenge with middle edge of type 1 colored 3 as any other piece would either contradict the minimality of x_2 or introduce an edge of type 0 and color 0 in R. Without loss of generality, one can thus assume that $x_0^{(0)} = 1$. Since $\partial_1^{(n)}$ has edges with colors 0 or 1, the upward triangular face containing $e^{(0)}$ would have colors $(0,0,0)$ or $(0,1,m)$ which would either imply that R has an edge of type 0 and color 0 or contradict the minimality of x_2. □

Remark 7.3.10 (Lozenges in R)**.** The two opposite 0 colored edges of a either 3 or m lozenge have the same type. Since Lemma 7.3.9 shows that 0 colored edges in R have type 1, the only orientations of 3 and m lozenges in R are such that the 3 edge has type 0 and the m edge has type 2, see Figure 7.14.

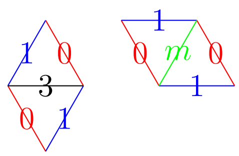

Figure 7.14: The two possible orientations of 3 and m lozenges in the region R.

We know from Lemma 7.3.9 that the color map C in R consists in triangular faces f such that $C(f) = (1,1,1)$ together with either 3 or m lozenges oriented as in Figure 7.14. In the rest of this section, we will view the color map C on R as a configuration of paths of color 0 from $R \cap \partial T[n_0, n_0, (0,0)]$ to $R \cap \partial_2 C$ as follows.
To each 3 or m lozenge of Figure 7.14, associate a line segment by joining the two centers of the opposite 0 colored edges. We define paths $(p_i, 1 \leqslant i \leqslant n_0)$ simultaneously. For each $1 \leqslant i \leqslant n_0$, the path p_i starts in the middle of the edge of type 1 in $R \cap \partial T[n_0, n_0, (0,0)]$ with origin $(n_0, 0) + i\xi^2$. At each step, a path having its endpoint with coordinate x_2 is continued by the affine line segment in the adjacent lozenge in R having an edge of color 0 with origin of coordinate $x_2 - 1$. Since this lozenge can only be one of the two lozenges of Figure 7.14, the paths are non-intersecting. After n_1 steps, endpoints are located in the middle of type 1 edges of color 0 on $R \cap \partial_2 C$. As there are n_0 such edges, the paths $(p_1, \ldots, p_{n_0})$ form a set of non-intersecting paths where for each $1 \leqslant i \leqslant n_0$, the path p_i has origin at $o_i = (n_0, 0) + (i - \frac{1}{2})\xi^2$ and target $t_i = o(e_i) + \frac{1}{2}\xi^5$ where $e_1, \ldots e_{n_0}$ are the 0 colored edges on $\partial_1 C$ ordered such that $h(e_1) > h(e_2) > \cdots > h(e_{n_0})$.
The paths $(p_i, 1 \leqslant i \leqslant n_0)$ can have two possible steps. We call the step induced by a m lozenge a horizontal step and the step induced by a 3 lozenge a vertical step in accordance with the red line segment joining the two opposite 0 colored edges in the lozenges of Figure 7.14.

Proposition 7.3.11 (Number of m edges in reduced color maps)**.** *Let C be a reduced color map. Denote by $n(m,R)$ and $n(3,R)$ the respective number of horizontal and vertical steps in R. Recall that $m(C)$ and $s(C)$ respectively denote the number of m lozenges and 3 lozenges in C. Then,*

$$m(C) = n(m,R) = G(C,1) \tag{7.3.8}$$

and

$$s(C) = n(3,R) = n_0 n_1 - G(C,1). \tag{7.3.9}$$

Proof. The only region in E_n where a reduced color map C has m colored edges is R. This amounts to count the number of horizontal steps in any path configuration $(p_i, 1 \leqslant i \leqslant n_0)$. Recall that $e_1, \ldots e_{n_0}$ are the 0 colored edges on $\partial_1 C$ ordered such that $h(e_1) > h(e_2) > \cdots > h(e_{n_0})$. As each path p_i goes from

$$o_i = (n_0, 0) + (i - \frac{1}{2})\xi^2 = \left(n_0 - \frac{1}{2}(i - \frac{1}{2}), (i - \frac{1}{2})\frac{\sqrt{3}}{2} \right)$$

to

$$\begin{aligned} t_i = o(e_i) + \frac{1}{2}\xi^5 &= (n,0) + (n - h(e_i))\xi^2 + \frac{1}{2}\xi^5 \\ &= \left(n - \frac{1}{2}(n - h(e_i) - \frac{1}{2}), \frac{\sqrt{3}}{2}(n - h(e_i) - \frac{1}{2}) \right), \end{aligned}$$

the number of vertical steps in p_i is given by

$$\frac{2}{\sqrt{3}} \cdot \left(\frac{\sqrt{3}}{2}(n - h(e_i) - \frac{1}{2}) - (i - \frac{1}{2})\frac{\sqrt{3}}{2} \right) = n - h(e_i) - i$$

so that the total number of vertical steps in the path configuration is

$$n(3,R) = \sum_{i=1}^{n_0} (n - h(e_i) - i).$$

Moreover, the number of 1 colored edges e' such that $h(e') < h(e_i)$ is given by

$$n(C, e_i) = n_1 - (n - h(e_i) - i)$$

so that

$$G(C,1) = \sum_{i=1}^{n_0} n(C, e_i) = \sum_{i=1}^{n_0} (n_1 - (n - h(e_i) - i)) = n_0 n_1 - n(3, R).$$

Therefore,

$$n(3, R) = n_0 n_1 - G(C, 1)$$

and, since each of the n_0 paths has a total number of steps given by n_1,

$$n(m, R) = n_0 n_1 - n(3, R) = G(C, 1).$$

□

Remark 7.3.12 (Number of reduced color maps)**.** One can derive the number of reduced color maps $N_{red}(n_0, n_1)$ since this number is equal to the number of paths configurations $(p_i, 1 \leqslant i \leqslant n_0)$ which can be computed by the determinantal formula of Lindström, Gessel and Viennot [Lin73], [GV85]:

$$N_{red}(n_0, n_1) = \det A \tag{7.3.10}$$

where $A = (a_{i,j}, 1 \leqslant i, j \leqslant n_0)$ is the matrix whose coefficients are given by

$$a_{i,j} = \binom{n_1}{n - h(e_j) - i} \tag{7.3.11}$$

with the convention that $a_{i,j} = 0$ if $h(e_j) + i \geqslant n$.

7.3.3 Proof of Theorem 7.1.5 in the case $G(C, 2) = 0$

Let C be a color map such that $G(C, 2) = 0$. Thanks to Proposition 7.3.7, one can reduce C to a reduced color map C' such that

$$m(C') = m(C) + n_0 n_1 - G(C, 0) \tag{7.3.12}$$

and

$$s(C') = s(C) - n_0 n_1 + G(C, 0) \tag{7.3.13}$$

Using (7.3.8) and (7.3.9) for the reduced color map C',

$$m(C') = G(C', 1) \tag{7.3.14}$$

$$s(C') = n_0 n_1 - G(C', 1). \tag{7.3.15}$$

Moreover, the reduction $C \mapsto C'$ does not change $\partial_1 C = \partial_1 C'$ so that $G(C, 1) = G(C', 1)$. Thus,

$$m(C) = G(C, 0) + G(C, 1) - n_0 n_1 \tag{7.3.16}$$

which is (7.1.2) since $G(C, 2) = 0$ and

$$s(C) = s(C') - G(C, 0) + n_0 n_1 = 2n_0 n_1 - G(C, 0) - G(C, 1) = n_0 n_1 - m(C) \tag{7.3.17}$$

which is (7.1.3).

7.4 The general case

In this section, we prove (7.1.2) by induction on $G(C,2)$. The case where $G(C,2)=0$ has been treated in Section 7.3.3. We first introduce a procedure in Section 7.4.1 which takes a color map C for which $G(C,2)\geqslant 1$ and transform it to a color map C' such that $G(C',2)=G(C,2)-1$. Using the previous transform, we finish the proof of Theorem 7.1.5 in Section 7.4.2.

7.4.1 Gash propagation

We introduce local configuration of two edges of the same type sharing a vertex called a gash, see Definition 7.4.1, which is inspired from gashes previously defined in [Buc+16], [Buc15] and [FT24]. Gashes will propagate across a color map C by local rules presented in Definition 7.4.2 until reaching some prescribed configuration or hitting $\partial_1^{(n)}$.

Definition 7.4.1 (Gash)**.** Let $x\in T_n$. A *gash* with center x is the union of the two edges $(x,x-\xi^{2l}),(x+\xi^{2l},x)$ for $l\in\{1,2\}$ with the data of

1. Original colors given by

$$C((x,x-\xi^{2l}))=1,\ C((x+\xi^{2l},x))=0 \text{ if } l=1$$
$$C((x,x-\xi^{2l}))=0,\ C((x+\xi^{2l},x))=1 \text{ if } l=2.$$

2. New colors given by replacing 0 with 1 and vice-versa in original colors.

The *type* of a gash is defined as the type $l\in\{1,2\}$ of its edges.

Let g be gash of type 2. The only possible values of the color map C adjacent to g are given by the configurations of Figure 7.15 that we label from (i) to (vi). The configurations can be rotated for type 1 gashes.

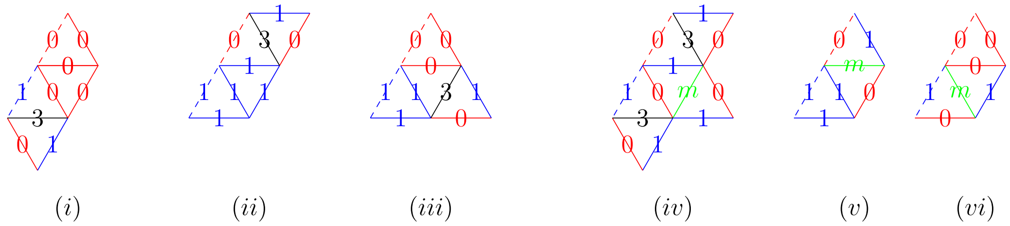

Figure 7.15: Possible adjacent configuration to a gash of type 2 in dashed edges. Only the original colors of gashes are represented.

Definition 7.4.2 (Gash propagation)**.** Let g be gash of type 2 with center x adjacent to a configuration $(i),(ii)$ or (iii). The *propagation* of g is the gash g' having center $x+\overline{\xi}$ (resp. $x+1$) in the case of configuration (i) (resp. (ii) or (iii)) together with the local replacement of Figure 7.16 depending on the adjacent configuration.

If g is adjacent to a configuration (iv), notice that there is a 0 opening at its center x and thus an arrow of color 0 at x with type 0. Reverting this arrow yields a configuration (i) adjacent to g and we define the propagation of g to be the gash g' of the same type as in step (1). Using a rotation, one defines propagations for type 1 gashes with the exception of configuration (iii) where the propagated gash is the gash of type 2 with center $x'=x+1$.

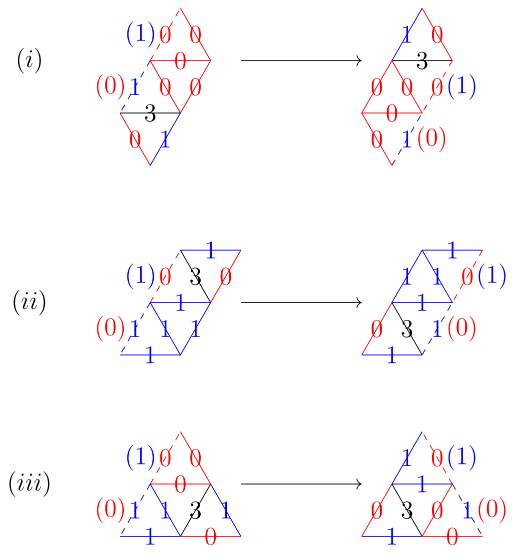

Figure 7.16: Propagation of a gash through configurations (i), (ii) and (iii). New colors of gashes are written in parenthesis.

Definition 7.4.3 (Propagation algorithm)**.** The *propagation algorithm* is the following algorithm.

Input: A color map C and a gash g of type $l \in \{1, 2\}$.

1. Set $g^{(0)} = g$, $x^{(0)} = x(g)$, $t^{(0)} = t(g)$.
2. **WHILE** $g^{(s)}$ is adjacent to $(i), (ii), (iii)$ or (iv): set $g^{(s+1)}$ to be the propagation of $g^{(s)}$ with center $x^{(s+1)}$ and type $t^{(s+1)}$.

Proposition 7.4.4 (Gash propagation)**.** *Let g be a gash of type 2 on $\partial_2^{(n)}$. The propagation algorithm terminates at a gash g' adjacent to configuration of type (v), (vi) or on $\partial_n^{(1)}$.*

Proof. One checks that propagations of Definition 7.4.2 do not change the type of the gash except in the case of a configuration (iii) which turns a gash of type 2 into a gash of type 1 and vice-versa. Since the starting gash g has type 2, the gashes have type either 1 or 2 along the propagation. At each step of the gash propagation, one has either $x_0^{(s+1)} < x_0^{(s)}$ or $x_1^{(s+1)} > x_1^{(s)}$ and $t^{(s)} \in \{1, 2\}$ which implies that the while loop terminates on a gash $g^{(\infty)}$ which is adjacent to a configuration (v) or (vi) or necessarily of type 1 on $\partial_n^{(1)}$. □

In the case where a gash is adjacent to a configuration (v) or (vi), one still wants to replace the original colors by the new ones. To do so, we introduce a local transformation called gash removal in Definition 7.4.5.

Definition 7.4.5 (Gash removal in (v) and (vi))**.** Let C be a color map and let g be a gash of type 2 with center x adjacent to a configuration (v) or (vi). The *removal* of g is the new color map C' defined by

$$\begin{aligned}
C'((x, x - \xi^4)) = 1, C'((x + \xi^4, x)) &= 0 \\
C'((x + 1, x)) = 1, C'((x, x - \xi^2)) &= 3 \text{ if } C((x + 1, x)) = m \\
C'((x + 1, x)) = 3, C'((x, x - \xi^2)) &= 0 \text{ if } C((x + 1, x)) = 0 \\
C'(e) &= C(e) \text{ otherwise.}
\end{aligned}$$

See Figure 7.17 for an illustration. Using rotation, one defines the gash removal for type 1 gashes adjacent to a configuration (v) or (vi).

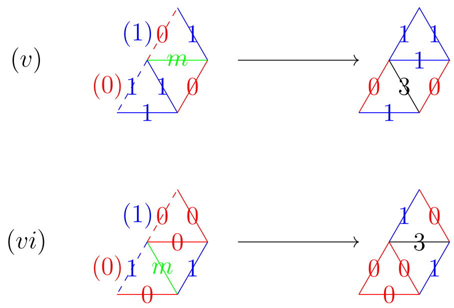

Figure 7.17: Gash removal in configurations (v) and (vi).

7.4.2 Proof of Theorem 7.1.5

We are now in position to prove Theorem 7.1.5 by induction on $G(C, 2)$. The case $G(C, 2) = 0$ has been treated in Section 7.3.3. Assume that the identities (7.1.2) and (7.1.3) hold for color maps C such that $G(C, 2) \leqslant N$ and consider a color map C such that $G(C, 2) = N + 1$.

Since $G(C, 2) \geqslant 1$, there exists a pair of edges $(x, x - \xi^4), (x + \xi^4, x) \in (\partial_n^{(2)})^2$ such that $C((x, x-\xi^4)) = 0,\ C((x+\xi^4, x)) = 1$ for some $x \in T_n$. Let g be the gash on $\partial_n^{(2)}$ with center x, original colors as above and new colors given by $C((x, x - \xi^4)) = 1,\ C((x + \xi^4, x)) = 0$ as in Definition 7.4.1. Applying the propagation algorithm of Definition 7.4.3 and using Proposition 7.4.4 yields a gash g' adjacent to a configuration (v), (vi) or on $\partial_n^{(1)}$. In the case of configurations (v) or (vi), apply the gash removal of Definition 7.4.5. In the case where $g' \in \partial_n^{(1)}$, replace the original colors by the new ones so that the 0 and 1 colors are swapped. Call C' the resulting color map. Then,

$$G(C', 2) = G(C, 2) - 1 = N \tag{7.4.1}$$
$$G(C', 0) = G(C, 0) \tag{7.4.2}$$

In the case where one used gash removal,

$$G(C', 1) = G(C, 1), \quad m(C') = m(C) - 1, \text{ and } s(C') = s(C) + 1. \tag{7.4.3}$$

whereas in the case where $g' \in \partial_n^{(1)}$,

$$G(C', 1) = G(C, 1) + 1, \quad m(C') = m(C), \text{ and } s(C') = s(C). \tag{7.4.4}$$

In both cases, applying the induction hypothesis to C' gives

$$m(C') = G(C', 0) + G(C', 1) + G(C', 2) - n_0 n_1 \tag{7.4.5}$$
$$s(C') = 2n_0 n_1 - G(C', 0) - G(C', 1) - G(C', 2) \tag{7.4.6}$$

which gives

$$m(C) = G(C, 0) + G(C, 1) + G(C, 2) - n_0 n_1 \tag{7.4.7}$$
$$s(C) = 2n_0 n_1 - G(C, 0) - G(C, 1) - G(C, 2) \tag{7.4.8}$$

as desired.

We finally state a Corollary of the main result which counts the number of faces having all of their edges of the same color, either 0 or 1.

Corollary 7.4.6 (Number of triangular pieces)**.** *Let C be a color map. Let $n^{(j)}, n_{(j)}, j \in \{0,1\}$ denote respectively the number of direct and reverse triangular faces $f \in F_n$ having all their edges of color j. Then,*

$$n^{(j)} = \frac{n_j(n_j+1)}{2} \text{ and } n_{(j)} = \frac{n_j(n_j-1)}{2}. \tag{7.4.9}$$

Proof. One can check that the gash propagation and removal steps preserve the number of faces f having edge colors $(0,0,0)$ or $(1,1,1)$. It therefore suffices to count them in a reduced color map which gives (7.4.9). □

Résumé en français

Chapitre 8

La théorie des matrices aléatoires

Cette thèse s'inscrit dans la théorie des matrices aléatoires, un domaine issu de l'analyse de données et des modèles statistiques pour les atomes lourds. Les origines de la théorie des matrices aléatoires remontent aux travaux de Wishart [Wis28] en statistique, puis à ceux de Wigner [Wig55], qui introduisit les matrices aléatoires dans le contexte de la mécanique quantique pour les atomes lourds. Wigner fut le premier à étudier les matrices de grande dimension, notamment dans le régime asymptotique où la dimension tend vers l'infini. Mehta a fortement influencé le domaine dans les années 1960, en posant les bases d'une théorie spectrale avec la première édition de son ouvrage [Meh04]. Le domaine s'est ensuite élargi avec les travaux de Marchenko et Pastur [MP68] en 1967, portant sur le spectre des matrices de covariance. Avec le temps, des techniques issues de l'analyse complexe, de la combinatoire et de la théorie du potentiel ont enrichi la théorie des matrices aléatoires. Une avancée majeure a été l'introduction des probabilités libres par Voiculescu [Voi86] dans les années 1980, permettant une meilleure compréhension du comportement asymptotique des grandes matrices aléatoires. Depuis les années 1990, le domaine a connu des progrès dans l'étude des valeurs propres extrêmes, des grandes déviations et des théorèmes limites pour les statistiques linéaires. Des questions d'universalité concernant le comportement des grandes matrices aléatoires ont émergé, certaines restant encore ouvertes à ce jour.

Ce chapitre a pour objectif de fournir un aperçu général de la théorie des matrices aléatoires. Il est composé de deux sections. La section 8.1 introduit les définitions fondamentales ainsi que plusieurs observables pertinents pour l'étude des matrices aléatoires. Elle présente également des exemples de matrices aléatoires, qui joueront un rôle central dans l'ensemble de cette thèse. La section 8.2 expose les deux principaux résultats de convergence des valeurs propres, concernant respectivement les matrices hermitiennes et les matrices à coefficients indépendants et identiquement distribués (i.i.d.), à savoir la loi semi-circulaire et la loi circulaire. Les principales références pour ce chapitre sont les ouvrages [Meh04; AGZ10; BS06; MS17], ainsi que les notes de cours [Spe20].

8.1 Matrices aléatoires

Cette section est une introduction aux matrices aléatoires avec des exemples de modèles qui interviennent dans cette thèse. Nous suivons principalement la référence [BS06].

8.1.1 Mesure spectrales de matrices aléatoires

Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace de probabilité. Pour tout $n \geqslant 1$, on note $\mathcal{M}_n(\mathbb{C})$ l'espace des matrices de taille $n \times n$ à coefficients complexes.

Definition 8.1.1 (Matrice aléatoire)**.** Une *matrice aléatoire* de taille $n \geqslant 1$ est une variable aléatoire $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ à valeurs dans $\mathcal{M}_n(\mathbb{C})$.

Une mesure naturelle associée à une matrice (non nécessairement aléatoire) est la mesure uniforme sur ses valeurs propres. Nous appelons cette mesure la distribution empirique des valeurs propres, ou la mesure spectrale, de la matrice.

Definition 8.1.2 (Distribution empirique des valeurs propres)**.** Soit $A \in \mathcal{M}_n(\mathbb{C})$ une matrice de taille $n \geqslant 1$. Notons $\lambda_1(A), \dots, \lambda_n(A)$ ses valeurs propres, comptées avec leurs multiplicités. La *distribution empirique des valeurs propres* de A est la mesure de probabilité $\mu_n(A)$ sur $\mathbb{C}$ définie par

$$\mu_n(A) \coloneqq \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k(A)} \ . \tag{8.1.1}$$

La distribution empirique des valeurs propres est supportée sur au plus n atomes distincts, situés aux valeurs propres de A, chacun ayant un poids proportionnel à la multiplicité de la valeur propre associée.

On définit $\mathcal{P}(\mathbb{C})$ comme l'espace des mesures de probabilité sur $\mathbb{C}$. Dans le cas où la matrice A_n est une matrice aléatoire de taille n, la mesure $\mu_n(A_n)$ est une variable aléatoire à valeurs dans $\mathcal{P}(\mathbb{C})$. On remarque que l'espace $\mathcal{P}(\mathbb{C})$ ne dépend pas de n, la taille de la matrice, ce qui permet de considérer les distributions empiriques des valeurs propres de matrices de tailles différentes comme des variables aléatoires appartenant au même espace.

8.1.2 Archétypes de matrices aléatoires

La définition d'une matrice aléatoire est générale car les coefficients peuvent avoir des corrélations et des lois de probabilité arbitraires. Nous verrons que le comportement des matrices aléatoires varie selon la dépendance structurelle de leurs coefficients. Nous présentons ici un aperçu des archétypes classiques de matrices aléatoires, comme exemples introductifs.

Matrices de Girko

L'exemple le plus simple de matrice aléatoire est celui où les coefficients sont des variables aléatoires indépendantes et identiquement distribuées (i.i.d.). De telles matrices sont appelées matrices de Girko, d'après les travaux de Girko [Gir18; Gir84].

Definition 8.1.3 (Matrice de Girko)**.** Soit $A = (a_{ij})_{i,j \geqslant 1}$ une famille de variables aléatoires i.i.d. La matrice $A_n = (a_{ij})_{1 \leqslant i,j \leqslant n}$ est appelée une *matrice de Girko* de taille n.

La définition 1.1.3 fixe les relations de dépendance dans la matrice aléatoire, mais reste très générale, car la loi commune des coefficients n'est pas précisée. Les premières occurrences de matrices à coefficients i.i.d. remontent aux travaux de Ginibre [Gin65], qui considérait le cas particulier où les coefficients suivent une loi normale complexe. Rappelons que la loi normale, ou gaussienne, de paramètres m et σ est définie comme la loi de probabilité ayant pour densité :

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

par rapport à la mesure de Lebesgue dx sur $\mathbb{R}$. On la note $\mathcal{N}(m, \sigma^2)$. On étend cette définition à $\mathbb{C}$ en prenant des parties réelle et imaginaire indépendantes.

Definition 8.1.4 (Loi normale complexe)**.** Soient Y et Z deux variables aléatoires réelles indépendantes suivant la loi $\mathcal{N}(0, \frac{1}{2})$. La *loi normale complexe* est la loi de la variable aléatoire $X = Y + iZ$. On la note $\mathcal{N}_{\mathbb{C}}(0, 1)$. On dit que X est une *gaussienne complexe standard.*

Une autre façon de décrire la loi normale complexe est de spécifier la densité $\frac{1}{\pi}\mathrm{e}^{-|z|^2}$ par rapport à la mesure de Lebesgue dz sur $\mathbb{C}$. On peut donc considérer des matrices de Girko dont les coefficients suivent une loi normale complexe. Ce cas particulier est appelé ensemble de Ginibre.

Definition 8.1.5 (Ensemble de Ginibre)**.** L'*ensemble de Ginibre* de taille $n \geqslant 1$ est la loi d'une matrice de Girko $A_n = (a_{ij})1 \leqslant i, j \leqslant n$ dont les coefficients a_{ij} sont des gaussiennes complexes standards. De manière équivalente, la loi de A_n sur $\mathcal{M}_n(\mathbb{C})$ est donnée par :

$$\mathrm{d}\mathbb{P}_n[A] \coloneqq \frac{1}{\pi^{n^2}} \exp\left(-\operatorname{Tr}\left[AA^*\right]\right) \mathrm{d}A\ , \tag{8.1.2}$$

où dA est la mesure de Lebesgue sur $\mathcal{M}_n(\mathbb{C})$ et A^* est l'adjoint de A.

L'ensemble de Ginibre possède une structure riche. C'est le premier modèle *intégrable* présenté ici, au sens où le fait de travailler avec la loi normale complexe permet d'effectuer de nombreux calculs explicites. Le premier exemple d'un tel calcul explicite est la loi jointe des valeurs propres d'une matrice de Ginibre, obtenue par Ginibre [Gin65].

Proposition 8.1.6 (Densité des valeurs propres de Ginibre, [Gin65])**.** *Soit A_n une matrice de Ginibre de taille $n \geqslant 1$. Alors, ses valeurs propres $(\lambda_1, \dots, \lambda_n)$ ont une loi jointe sur $\mathbb{C}^n$ donnée par :*

$$\frac{1}{Z_n} \prod_{i<j} |\lambda_i - \lambda_j|^2 \mathrm{e}^{-\sum_{i=1}^n |\lambda_i|^2} \mathrm{d}\lambda\ , \tag{8.1.3}$$

où dλ *est la mesure de Lebesgue sur $\mathbb{C}^n$ et où Z_n est une constante de normalisation.*

Matrices de Wigner

La théorie des matrices aléatoires a joué un rôle fondamental dans le développement de la mécanique quantique entre 1940 et 1950. Dans ce contexte, la motivation était de modéliser les noyaux lourds à l'aide d'un hamiltonien discrétisé représenté par une matrice. L'hamiltonien discrétisé satisfait certaines hypothèses de symétrie ce qui impose une condition d'hermiticité à la matrice discrétisée. Il est donc naturel de considérer des matrices aléatoires hermitiennes. De telles matrices sont appelées matrices de Wigner, d'après les travaux de Wigner [Wig55]. Pour $n \geqslant 1$, on note $\mathcal{H}_n$ l'espace des matrices hermitiennes de taille n.

Definition 8.1.7 (Matrice de Wigner)**.** Soit $A = (a_{ij})_{1 \leqslant i \leqslant j}$ une famille de variables aléatoires indépendantes et soit $n \geqslant 1$. On appelle *matrice de Wigner* de taille n la matrice $A_n = (a_{ij})_{1 \leqslant i, j \leqslant n}$ telle que pour $i \geqslant j$, on ait $a_{ij} = \overline{a_{ji}}$.

Comme dans la définition 1.1.3, la définition d'une matrice de Wigner ne précise pas la loi de probabilité des éléments situés au-dessus de la diagonale. L'exemple le plus célèbre est celui où les coefficients strictement au-dessus de la diagonale sont des gaussiennes complexes standards, et ceux sur la diagonale sont des gaussiennes réels standards. Ce cas particulier est appelé ensemble unitaire gaussien (GUE).

Definition 8.1.8 (Ensemble unitaire gaussien)**.** L'*ensemble unitaire gaussien (GUE)* de taille $n \geqslant 1$ est la loi de la matrice de Wigner dont les coefficients a_{ij} pour $j > i$ suivent la loi $\mathcal{N}_{\mathbb{C}}(0,1)$, et dont les éléments diagonaux a_{ii} sont des gaussiennes réelles standards indépendants. De manière équivalente, la loi de A_n sur $\mathcal{H}_n$ est donnée par :

$$\mathrm{d}\mathbb{P}_n[A] := \frac{1}{Z_n} \exp\left(-\frac{1}{2} \operatorname{Tr}\left[A^2\right]\right) \mathrm{d}A \tag{8.1.4}$$

où $\mathrm{d}A$ est la mesure de Lebesgue sur $\mathcal{H}_n$.

Le nom de cet ensemble vient du fait que la loi d'une matrice du GUE est invariante par conjugaison unitaire. L'ensemble GUE peut être vu comme l'analogue hermitien de l'ensemble de Ginibre. En effet, si A est une matrice de Ginibre, alors $X = \frac{A+A^*}{\sqrt{2}}$ est une matrice du GUE. Inversement, si X et Y sont deux matrices du GUE indépendantes, alors $A = \frac{1}{\sqrt{2}}X + i\frac{1}{\sqrt{2}}Y$ est une matrice de Ginibre.

Comme dans le cas de l'ensemble de Ginibre, la loi jointe des valeurs propres d'une matrice du GUE est connue. On notera que la condition d'hermiticité implique que les valeurs propres sont réelles.

Proposition 8.1.9 (Loi des valeurs propres du GUE [AGZ10])**.** *Soit A_n une matrice GUE de taille $n \geqslant 1$. Alors ses valeurs propres $(\lambda_1, \dots, \lambda_n)$ ont une loi jointe donnée par :*

$$\frac{1}{Z_n} \prod_{i<j} |\lambda_i - \lambda_j|^2 \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^n \lambda_i^2} \mathrm{d}\lambda \tag{8.1.5}$$

où $\mathrm{d}\lambda$ est la mesure de Lebesgue sur $\mathbb{R}^n$ et Z_n est une constante de normalisation.

Matrices unitaires

Pour $n \geqslant 1$, considérons le groupe unitaire

$$\mathrm{U}(n) = \{U \in \mathcal{M}_n(\mathbb{C}) \mid UU^* = U^*U = \mathrm{id}_{\mathbb{C}^n}\}.$$

C'est un sous-groupe topologique compact de $\mathcal{M}_n(\mathbb{C})$, ce qui signifie que, en plus de sa structure de groupe, $\mathrm{U}(n)$ est un espace topologique pour lequel la multiplication matricielle $\cdot : \mathrm{U}(n) \times \mathrm{U}(n) \to \mathrm{U}(n),\ (x,y) \mapsto x \cdot y$ et l'inversion ${}^{-1} : \mathrm{U}(n) \to \mathrm{U}(n),\ x \mapsto x^{-1}$ sont des applications continues. L'étude des matrices unitaires aléatoires a été initiée par Dyson [Dys62], qui a étudié des cas intégrables appelés ensembles circulaires, voir aussi les travaux de Girko [Gir85].
Sur les groupes topologiques, on dispose de la notion de mesure de Haar qui est l'analogue de la mesure uniforme. Nous suivons ici l'approche du chapitre 5 de [Far08].

Definition 8.1.10 (Mesure invariante à gauche)**.** Soit G un groupe localement compact. Une mesure de Radon $\mu \geqslant 0$ sur G est dite *invariante à gauche* si, pour tout $h \in G$ et toute fonction continue f sur G à support compact,

$$\int_G f(hg)\mu(\mathrm{d}g) = \int_G f(g)\mu(\mathrm{d}g)$$

Nous énonçons le théorème principal du chapitre 5 de [Far08], qui affirme qu'il existe une unique mesure invariante à gauche sur G, à un facteur positif près. Cette mesure est appelée mesure de Haar à gauche sur G.

Theorem 8.1.11 (Existence de la mesure de Haar, [Far08])**.** *Tout groupe localement compact admet une mesure invariante à gauche. De plus, une telle mesure est unique à un facteur multiplicatif près.*

Nous désignerons par *la* mesure de Haar sur G la mesure invariante à gauche μ sur G dont la masse totale vaut un, c'est-à-dire telle que $\mu(G) = 1$. Revenons au groupe unitaire $G = \mathrm{U}(n)$. On dispose d'une construction explicite de la mesure de Haar comme suit. Soit A_n une matrice de Ginibre de taille n. On applique à A_n la procédure d'orthonormalisation de Gram-Schmidt. Alors, la loi de la matrice unitaire ainsi obtenue, dont les colonnes sont orthonormées, est la mesure de Haar. L'invariance par multiplication par des matrices de permutation, qui sont des cas particuliers de matrices unitaires, implique que les coefficients d'une matrice de Haar ont tous la même loi.

Les valeurs propres d'une matrice unitaire sont situées sur le cercle unité

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\} \ .$$

Pour une matrice aléatoire U suivant la mesure de Haar, il est naturel de s'intéresser à la loi jointe de ses valeurs propres. Nous paramétrons ces valeurs propres par leurs angles, de sorte que $\lambda_k = \mathrm{e}^{i\theta_k}$ pour $1 \leqslant k \leqslant n$ et $\theta_k \in [0, 2\pi)$. La densité jointe peut être calculée explicitement, et l'on renvoie à la référence [HP00] pour une démonstration.

Theorem 8.1.12 (Densité jointe des valeurs propres des matrices unitaires de Haar, [HP00])**.** *Pour $n \geqslant 1$ et U_n distribué selon la mesure de Haar sur $\mathrm{U}(n)$, la loi jointe de $(\theta_1, \ldots, \theta_n)$ admet la densité*

$$f(\theta_1, \ldots, \theta_n) = \frac{1}{Z_n} \prod_{j<\ell} |\mathrm{e}^{i\theta_j} - \mathrm{e}^{i\theta_\ell}|^2. \tag{8.1.6}$$

Il existe une généralisation de (8.1.6) à un paramètre $\beta > 0$. Ces lois sont appelées ensembles circulaires β, voir [DG04].

Definition 8.1.13 (Ensemble circulaire β)**.** Fixons $\beta > 0$ et $n \geqslant 1$. L'*ensemble circulaire* β est la distribution de probabilité sur $[0, 2\pi)^n$ donnée par

$$\mathrm{d}\mathbb{P}_{\beta,n}[\theta] \coloneqq \frac{1}{Z_{\beta,n}} \prod_{j<\ell} |\mathrm{e}^{i\theta_j} - \mathrm{e}^{i\theta_\ell}|^\beta \mathrm{d}\theta. \tag{8.1.7}$$

Pour $\beta = 2$, on retrouve la distribution des valeurs propres des matrices unitaires aléatoires de Haar, également appelée ensemble unitaire circulaire. Pour $\beta = 1$ et $\beta = 4$, la distribution (8.1.7) correspond respectivement aux valeurs propres de matrices aléatoires de Haar dans les groupes orthogonal et symplectique.

Matrices de permutation

Cette section traite des matrices de permutation aléatoires, qui constituent des cas particuliers de matrices orthogonales aléatoires. Pour $n \geqslant 1$, notons S_n le groupe des permutations de $[n] = \{1, \ldots, n\}$. Pour une permutation $\sigma \in S_n$, sa matrice de permutation associée est $A = A(\sigma) = (a_{ij})_{1 \leqslant i,j \leqslant n}$, où pour $1 \leqslant i, j \leqslant n$,

$$a_{ij} \coloneqq \mathbb{1}_{\sigma(j)=i}.$$

On appelle matrice de permutation toute matrice qui peut s'écrire comme la matrice associée à une permutation. La décomposition en cycles d'une permutation $\sigma \in S_n$ est le vecteur $(C_k)_{1 \leqslant k \leqslant n}$ où $C_k = C_k(\sigma)$ est le nombre de cycles de longueur k dans σ.
Pour $k \geqslant 1$, soit $B_k \in \mathcal{M}_k(\mathbb{C})$ la matrice $k \times k$ associée à la permutation ayant un seul cycle $(1 \cdots k)$:

$$B_k = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Toute matrice de permutation $A = A(\sigma)$ pour un certain $\sigma \in S_n$ est conjuguée à une matrice par blocs comportant $C_k(\sigma)$ blocs B_k pour chaque $1 \leqslant k \leqslant n$. Par conséquent, les valeurs propres d'une matrice de permutation $A(\sigma)$ sont les racines de l'unité

$$\mathrm{e}^{\frac{2i\pi\ell}{k}}, \quad 0 \leqslant \ell \leqslant k-1$$

apparaissant chacune avec une multiplicité $C_k(\sigma)$.

Une mesure définie sur l'ensemble des permutations induit naturellement une mesure sur l'ensemble des matrices de permutation via l'application $\sigma \mapsto A(\sigma)$. Nous introduisons ici une mesure importante sur S_n, appelée loi d'Ewens, introduite dans [Ewe72].

Definition 8.1.14 (Loi d'Ewens)**.** La *loi d'Ewens* de paramètre $\theta > 0$ est la mesure de probabilité $\mathrm{d}\mathbb{P}_\theta$ sur S_n définie par

$$\mathrm{d}\mathbb{P}_\theta^{(n)}[\sigma] := \frac{\theta^{|\sigma|}}{Z_\theta^{(n)}}, \tag{8.1.8}$$

où $|\sigma| = \sum_{k=1}^n C_k(\sigma)$ désigne le nombre total de cycles dans la permutation σ, et $Z_\theta^{(n)}$ est une constante de normalisation.

Lorsque $\theta = 1$, la loi d'Ewens coïncide avec la mesure uniforme sur S_n. Il s'agit d'un exemple de mesure centrale, c'est-à-dire une mesure constante sur chaque classe de conjugaison de S_n, les longueurs de cycles étant invariantes par conjugaison.

8.2 Convergence des mesures spectrales

Cette section présente des résultats de convergence des mesures spectrales lorsque la dimension des matrices tend vers l'infini. Les premières motivations pour considérer de tels régimes limites remontent aux travaux de Wigner [Wig55; Eug58] dans le contexte de la mécanique quantique. Wigner a démontré que, sous une normalisation appropriée, la distribution empirique des valeurs propres des matrices du GUE converge en moyenne vers une loi limite, appelée loi du demi-cercle, voir le Théorème 8.2.1 ci-dessous. Ce résultat a ensuite été généralisé par Arnold [Arn71; Arn67]. Dans le cas des matrices aléatoires de type Girko, l'analogue de la loi du demi-cercle est la loi circulaire, qui correspond à la loi uniforme sur le disque unité du plan complexe. La convergence en moyenne de la distribution empirique des valeurs propres a d'abord été établie par Mehta [Meh67] pour l'ensemble de Ginibre. Edelman a ensuite traité le cas particulier où les entrées sont des variables gaussiennes réelles. Silverstein a étendu ces résultats en prouvant la convergence presque sûre pour les matrices de Ginibre. Ces premiers travaux s'appuyaient fortement sur les formules explicites de densité jointe des valeurs propres dans le cas gaussien. Une approche plus générale, visant à étendre ces résultats à des distributions non gaussiennes, a été initiée par Girko [Gir84] puis développée par Bai [Bai97]. La version la plus générale a finalement été obtenue par Tao et Vu [TV08], en collaboration avec Krishnapur [TVK10].

8.2.1 Convergence de mesures aléatoires

Rappelons que la distribution empirique des valeurs propres d'une matrice A_n de valeurs propres $(\lambda_i)_{1\leqslant i\leqslant n}$ est la mesure de probabilité

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i}.$$

On munit l'espace $\mathcal{P}(\mathbb{C})$ des mesures de probabilité sur $\mathbb{C}$ de la topologie de la convergence faible, définie par rapport aux fonctions continues et bornées. La convergence faible d'une suite $(\mu_n)_{n\geqslant 1}$ vers une mesure limite μ sera notée $\mu_n \implies \mu$. Pour des mesures aléatoires, on dit que μ_n converge faiblement vers μ presque sûrement si

$$\mu_n \implies \mu \quad \text{presque sûrement},$$

ce qui signifie que, avec probabilité un, pour toute fonction continue et bornée f,

$$\int f\mathrm{d}\mu_n \to \int f\mathrm{d}\mu.$$

Pour une mesure aléatoire μ, on note $\mathbb{E}[\mu]$ la mesure de probabilité définie par

$$\mathbb{E}[\mu](B) = \mathbb{E}[\mu(B)]$$

pour tout borélien B. Dans le cas où μ est la distribution empirique des valeurs propres, la mesure $\mathbb{E}[\mu]$, qui est une mesure déterministe, est appelée la distribution moyenne des valeurs propres.

8.2.2 Matrices de Wigner et loi du demi-cercle

Dans cette section, nous nous concentrons sur les matrices de Wigner telles que définies en 8.1.7. Nous présentons une version générale du résultat de convergence de la distribution empirique des valeurs propres pour ces matrices que l'on peut trouver dans [BS06].

Theorem 8.2.1 (Loi du demi-cercle de Wigner, [BS06])**.** *Soit A_n une matrice de Wigner de taille n telle que les entrées au-dessus de la diagonale aient une variance unitaire. On suppose que toutes les entrées sont centrées. Alors, presque sûrement,*

$$\mu_n\left(\frac{1}{\sqrt{n}}A_n\right) \implies \mu_{s.c}$$

où

$$\mu_{s.c}(\mathrm{d}x) := \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbb{1}_{|x|<2}\mathrm{d}x.$$

La mesure limite $\mu_{s.c}$ est appelée la loi du demi-cercle, voir Figure 8.1. Le Théorème 8.2.1 constitue le premier exemple d' *universalité*, puisqu'il est valable pour toute paire de lois, pour la diagonale et hors-diagonale, tant que cette dernière a une variance unitaire.

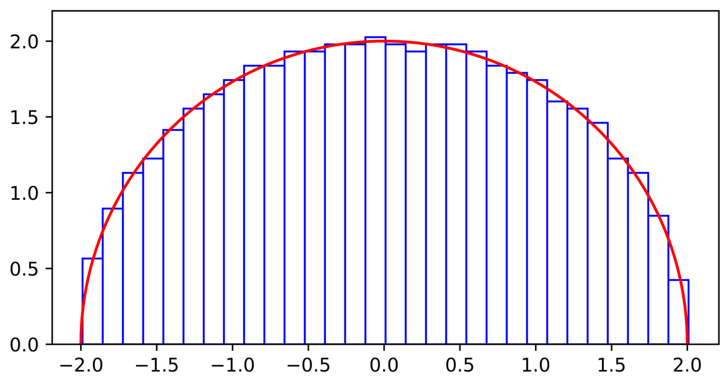

Figure 8.1: Illustration du Théorème 8.2.1. Valeurs propres d'une matrice GUE normalisée de taille 10^3. La densité de la loi du demi-cercle est tracée en rouge.

Remark 8.2.2. Donnons quelques remarques :

1. Dans le cas où les entrées hors diagonale ont une variance générale $\sigma^2 > 0$, la loi limite obtenue est la loi du demi-cercle dilatée, de densité

$$\frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\mathbb{1}_{|x|<2\sigma}\mathrm{d}x.$$

2. L'hypothèse d'identicité des lois des entrées peut être relâchée. Considérons une matrice de Wigner dont les entrées diagonales et hors diagonales sont indépendantes mais pas nécessairement identiquement distribuées et où la loi de chaque entrée peut dépendre de n. Si la condition suivante est vérifiée :

$$\forall \eta > 0 \lim_{n\to\infty} \frac{1}{n^2} \sum_{j,k=1}^{n} \mathbb{E}\left[|a_{jk}^{(n)}|^2 \mathbb{1}_{|a_{jk}^{(n)}|\geqslant \eta\sqrt{n}}\right],$$

alors la conclusion du Théorème 8.2.1 reste valable.

3. La condition sur le moment d'ordre deux des entrées hors-diagonales est à la fois nécessaire et suffisante dans le Théorème 8.2.1. Dans le cas d'entrées à queues lourdes, la distribution empirique des valeurs propres converge vers d'autres lois dépendant du paramètre de la loi stable. Nous renvoyons à [BC94], [BDG09] et [BCC11] pour plus de détails.

8.2.3 Matrices de Girko et loi circulaire

Cette section présente l'analogue du théorème de Wigner pour les matrices de Girko, c'est-à-dire les matrices à coefficients i.i.d. sans condition d'hermiticité. La version générale suivante peut être trouvée dans [TVK10], voir aussi [BC12] pour plus de détails.

Theorem 8.2.3 (Loi circulaire, [TVK10])**.** *Soit $A_n = (a_{ij})_{1\leqslant i,j\leqslant n}$ où $(a_{ij})_{i,j\geqslant 1}$ sont des variables aléatoires i.i.d. telles que $\mathbb{E}[a_{ij}] = 0$ et $\mathbb{E}[|a_{ij}|^2] = 1$. Alors, presque sûrement,*

$$\mu_n\left(\frac{1}{\sqrt{n}}A_n\right) \implies \mu_c, \tag{8.2.1}$$

où

$$\mu_c(\mathrm{d}z) \coloneqq \frac{1}{\pi}\mathbb{1}_{|z|<1}\mathrm{d}z.$$

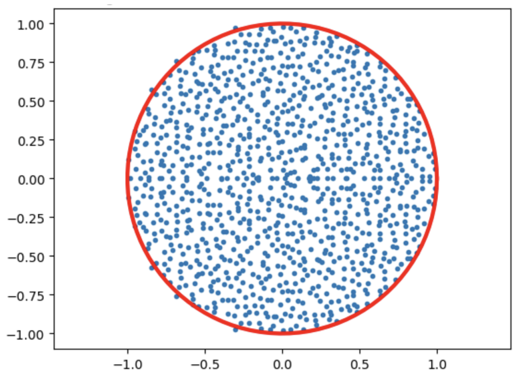

Figure 8.2: Illustration du Théorème 8.2.3. Valeurs propres d'une matrice de Ginibre normalisée de taille 500. Le cercle unité est représenté en rouge.

La mesure de probabilité $\mu_c(\mathrm{d}z) = \frac{1}{\pi}\mathbb{1}_{|z|<1}\mathrm{d}z$ est appelée la loi circulaire, ou loi du cercle. Elle peut être vue comme l'analogue non hermitien de la loi du demi-cercle. La démonstration du Théorème 8.2.3 repose sur la technique d'hermitisation, introduite par Girko [Gir84]. Nous renvoyons à [BC12] et [TVK10; TV08] pour l'utilisation de cette technique afin d'obtenir le Théorème 8.2.3.

Chapitre 9

Polynômes caractéristiques

Ce chapitre est consacré aux polynômes caractéristiques de matrices aléatoires. Contrairement au chapitre précédent, qui étudie les valeurs propres via la mesure spectrale, cette approche considère le polynôme caractéristique comme une fonction aléatoire. Sous une normalisation appropriée, l'objectif est d'établir sa convergence vers une fonction analytique limite.

L'étude des polynômes caractéristiques en théorie des matrices aléatoires est double. D'une part, il est possible de partir d'un modèle de matrice aléatoire donné et d'établir la convergence de son polynôme caractéristique. Cela permet d'obtenir des informations sur les propriétés spectrales de la matrice. Au-delà de l'analyse spectrale, les limites de ces polynômes font intervenir des fonctions aléatoires remarquables. Par exemple, dans le cas des matrices de Girko centrées, la fonction limite est l'exponentielle d'une fonction analytique gaussienne sur le plan complexe qui a été étudiée indépendamment, voir par exemple [Hou+09].

Une seconde approche consiste à utiliser les polynômes caractéristiques pour reformuler des problèmes existants en les reliant à des matrices aléatoires bien choisies. Cette méthode a été employée par Keating et Snaith [KS00], qui ont établi un lien entre le polynôme caractéristique des matrices unitaires de Haar et la fonction Zêta de Riemann. Leur travail a conduit à une conjecture sur les moments de la fonction Zêta, motivée par la conjecture de Montgomery [Mon73] en théorie analytique des nombres et les travaux de Rudnick et Sarnak [RS96] sur les liens entre les zéros des fonctions L et les matrices aléatoires. Les polynômes caractéristiques apparaissent également dans d'autres contextes, notamment en physique statistique où ils sont reliés aux gaz à corrélation logarithmique et aux champs gaussiens, voir [BK22] pour ces derniers aspects.

9.1 Polynômes caractéristiques aléatoires et traces

L'objet central de ce chapitre est le polynôme caractéristique

$$p_n(z) \coloneqq \det(I_n - zA_n)$$

d'une matrice aléatoire A_n. Nous cherchons à étudier sa convergence en tant que variable aléatoire à valeurs dans l'espace des fonctions holomorphes, muni de la topologie de la convergence uniforme locale. Les coefficients $(c_k^{(n)})_{0\leqslant k\leqslant n}$ de p_n, définis par

$$p_n(z) = \sum_{k=0}^{n} c_k^{(n)} z^k,$$

sont appelés les coefficients séculiers. Ils sont reliés aux traces des puissances de A_n via

$$c_k^{(n)} = P_k\left(\mathrm{Tr}[A_n], \dots, \mathrm{Tr}\left[A_n^k\right]\right)$$

où $(P_k)_{k\geqslant 0}$ est une famille de polynômes indépendante de n. Ainsi, l'étude des coefficients séculaires et des polynômes caractéristiques peut se ramener à celle de la convergence des traces $(\mathrm{Tr}[A_n^k])_{k\geqslant 1}$. On peut également mettre en évidence ce lien avec les traces en développant le logarithme :

$$\log p_n(z) = -\sum_{k\geqslant 1} \frac{z^k}{k} \mathrm{Tr}\left[A_n^k\right] \tag{9.1.1}$$

comme identité formelle. Si l'on montre la convergence jointe des traces dans (9.1.1) vers une certaine famille de coefficients, un candidat naturel pour la fonction limite est la fonction analytique ayant cette famille comme coefficients. Ce point de vue a été utilisé dans divers contextes, voir par exemple [BCG22; Cos23; CLZ24; NPS23]. Comme nous le verrons dans les résultats ci-dessous, les fonctions limites des polynômes caractéristiques de grandes matrices présentent souvent une structure impliquant l'exponentielle d'une série entière aléatoire.

9.2 Contributions

Nous présentons ici nos contributions concernant la convergence des polynômes caractéristiques pour deux modèles intégrables, décrits dans les Sections 9.2.1 et 9.2.2, correspondant aux articles [FG23] et [Fra25], qui font respectivement l'objet des Chapitres 4 et 5.

9.2.1 Polynôme caractéristique de matrices elliptiques gaussiennes

Les matrices aléatoires que nous considérons dans cette section sont issues de l'ensemble de Ginibre elliptique complexe, introduit par Girko dans [Gir86]. Ce modèle est paramétré par un réel $t \in [0,1]$ et interpole entre l'ensemble de Ginibre (Définition 8.1.5) et l'ensemble unitaire gaussien (Définition 8.1.8) pour $t = 0$ et $t = 1$ respectivement. Sa loi est celle d'une matrice aléatoire construite comme suit : Soient X_n et Y_n deux matrices aléatoires indépendantes de l'ensemble unitaire gaussien de taille $n \geqslant 1$. La loi de l'ensemble de Ginibre elliptique au paramètre $t \in [0,1]$ est celle de la matrice

$$A_{n,t} = \sqrt{\frac{1+t}{2}} X_n + i\sqrt{\frac{1-t}{2}} Y_n, \tag{9.2.1}$$

où i est le complexe de module un et de phase $\frac{\pi}{2}$. De manière équivalente, $A_{n,t}$ admet une densité proportionnelle à

$$\exp\left(-\frac{1}{1-t^2}\mathrm{Tr}\left[M^*M - \frac{t}{2}\left(M^2 + (M^*)^2\right)\right]\right) \mathrm{d}M, \tag{9.2.2}$$

où $\mathrm{d}M = \prod_{1\leqslant i,j\leqslant n} \mathrm{d}M_{ij}$ est la mesure de Lebesgue produit sur les coefficients de la matrice. La distribution limite des valeurs propres a été obtenue par Girko [Gir86]. Il s'agit de la loi uniforme sur l'ellipse

$$\mathcal{E}_t := \left\{x + iy \in \mathbb{C} \mid \left(\frac{x}{1+t}\right)^2 + \left(\frac{y}{1-t}\right)^2 \leqslant 1\right\}.$$

On définit la fonction $f_{n,t} : \mathbb{D} \to \mathbb{C}$ comme le polynôme caractéristique normalisé de $A_{n,t}$,

$$f_{n,t}(z) := \det\left(1 + tz^2 - \frac{z}{\sqrt{n}} A_{n,t}\right) \mathrm{e}^{-\frac{ntz^2}{2}}. \tag{9.2.3}$$

On munit l'espace des fonctions holomorphes sur $\mathbb{D}$ de la topologie de la convergence uniforme sur les compacts. Notre résultat principal est la convergence suivante.

Theorem 9.2.1 (Convergence du polynôme caractéristique normalisé)**.** *On a la convergence en loi, pour la topologie de la convergence locale uniforme,*

$$f_{n,t} \xrightarrow[n\to\infty]{\text{loi}} \exp(-F_t)$$

où F_t est une fonction holomorphe gaussienne sur $\mathbb{D}$ définie par

$$F_t(z) := \sum_{k \geqslant 1} X_k \frac{z^k}{\sqrt{k}} \tag{9.2.4}$$

pour une famille $(X_k)_{k\geqslant 1}$ de variables aléatoires gaussiennes complexes indépendantes, vérifiant

$$\mathbb{E}[X_k] = 0, \quad \mathbb{E}[X_k^2] = t^k \quad \textit{et} \quad \mathbb{E}[|X_k|^2] = 1.$$

En particulier, pour $t = 1$, le Théorème 9.2.1 montre que le polynôme caractéristique des matrices de type GUE, convenablement normalisé, converge vers une fonction holomorphe aléatoire. À partir de ce théorème, on déduit l'absence de valeurs propres hors de l'ellipse, ce qui est l'analogue elliptique de la convergence du rayon spectral des matrices de Girko obtenue dans [BCG22].

Corollary 9.2.2 (Absence d'outliers)**.** *Soit $C \subset \mathbb{C}$ un ensemble fermé disjoint de $\mathcal{E}_t$. Alors, lorsque $n \to \infty$, nous avons la convergence en probabilités*

$$N_n(C) := \#\left\{i \in [n] : \frac{\lambda_i}{\sqrt{n}} \in C\right\} \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{9.2.5}$$

On s'attend à ce que la convergence du Théorème 9.2.1 s'étende dans un cadre beaucoup plus général, comme conjecturé dans [BCG22], voir la Section 9.3.1. La limite dépendrait uniquement des quatre premiers moments des coefficients de la matrice aléatoire. Un aperçu de cette universalité peut être observé lorsqu'on calcule l'espérance du polynôme caractéristique, qui dépend uniquement de $t = \mathbb{E}[a_{12}a_{21}]$. On a alors la convergence suivante pour l'espérance du polynôme caractéristique des matrices elliptiques.

Theorem 9.2.3 (Polynôme caractéristique moyen)**.** *Soit $A_{n,t} = (a_{ij}, 1 \leqslant i, j \leqslant n)$ une matrice aléatoire telle que $\{(a_{ij}, a_{ji}), 1 \leqslant i < j \leqslant n\}$ soient des paires i.i.d. centrées et indépendantes de la famille i.i.d. centrée $\{a_{ii}, 1 \leqslant i \leqslant n\}$, avec $\mathbb{E}[|a_{ij}|^2] < \infty$ pour tout $1 \leqslant i, j \leqslant n$ et $\mathbb{E}[a_{12}a_{21}] = t \in [0, 1]$. Alors, pour z uniformément dans $\mathbb{D}$,*

$$\lim_{n\to+\infty} \mathbb{E}\left[\det\left(1 + tz^2 - \frac{z}{\sqrt{n}} A_{n,t}\right) \mathrm{e}^{-\frac{ntz^2}{2}}\right] = \frac{1}{\sqrt{1 - tz^2}}. \tag{9.2.6}$$

9.2.2 Polynôme caractéristique de matrices d'Ewens

Dans cette section, nous considérons des matrices de permutation distribuées selon la mesure d'Ewens généralisée, introduite par Nikeghbali et Zeindler [NZ13], qui généralise la mesure d'Ewens (8.1.8). Rappelons que pour une permutation $\sigma \in S_n$ et $k \geqslant 1$, $C_k(\sigma)$ est le nombre de cycles de σ de longueur k.

Definition 9.2.4 (Mesure d'Ewens généralisée, [NZ13])**.** Soit $\Theta = (\theta_k)_{k\geqslant 1}$ une suite de réels strictement positifs. Pour $n \geqslant 1$, la *mesure d'Ewens généralisée* est la mesure de probabilité $\mathrm{d}\mathbb{P}_n^{\Theta}$ sur S_n définie par

$$\mathrm{d}\mathbb{P}_n^{\Theta}[\sigma] := \frac{1}{n! h_n^{\Theta}} \prod_{k=1}^{n} \theta_k^{C_k(\sigma)}. \tag{9.2.7}$$

À partir de la suite $\Theta = (\theta_k)_{k\geqslant 1}$, on définit la série formelle comme dans [NZ13],

$$g_{\Theta}(z) := \sum_{k\geqslant 1} \frac{\theta_k}{k} z^k \text{ et } G_{\Theta}(z) := \exp(g_{\Theta}(z)) \tag{9.2.8}$$

Pour $n \geqslant 1$ et $\Theta = (\theta_k)_{k\geqslant 1}$ comme ci-dessus, on considère A_n la matrice aléatoire associée à une permutation σ distribuée selon (9.2.7). Considérons son polynôme caractéristique

$$p_n(z) = \det(1 - zA_n) \tag{9.2.9}$$

dans le disque unité $z \in \mathbb{D} = \{x \in \mathbb{C} : |x| < 1\}$. Notons $\mathcal{H}(\mathbb{D})$ l'espace des fonctions holomorphes sur $\mathbb{D}$, muni de la topologie de la convergence uniforme sur les compacts de $\mathbb{D}$. Notre résultat principal est la convergence de p_n, en tant que variable aléatoire dans $\mathcal{H}(\mathbb{D})$, en loi vers une fonction limite $F \in \mathcal{H}(\mathbb{D})$. Cette convergence a lieu pour des paramètres Θ tels que la série génératrice g_{Θ} vérifie certaines conditions que nous définissons ci-après. Il s'agit d'une adaptation d'une définition donnée dans la Section 5.2.1 de [Hwa94]. On peut également la retrouver en tant que Définition 2.9 dans [Hug+13] ou Définition 2.8 dans [NZ13].

Definition 9.2.5 (Fonction de classe logarithmique)**.** Une fonction g est dite appartenir à $F(r, \gamma, K)$ pour $r > 0$, $\gamma \geqslant 0$ et $K \in \mathbb{C}$ si :

- Il existe $R > r$ et $\phi \in (0, \pi/2)$ tels que g est holomorphe dans $\Delta(r, R, \phi) \setminus r$ où $\Delta(r, R, \phi) = \{z \in \mathbb{C} : |z| \leqslant R, |\arg(z - r)| \geqslant \phi\}$.
- Lorsque $z \to r$, $g(z) = -\gamma \log(1 - z/r) + K + O(z - r)$.

Dans le cas de la mesure d'Ewens (8.1.8) de paramètre θ, on a $g_{\Theta}(z) = -\theta \log(1 - z)$ et $g_{\Theta} \in F(1, \theta, 0)$. Notons que si $\gamma > 0$, alors le paramètre $r > 0$ est unique.

Notre résultat principal est le Théorème 9.2.6, qui donne la convergence du polynôme caractéristique vers une fonction limite pour les suites Θ telles que g est de classe logarithmique.

Theorem 9.2.6 (Convergence du polynôme caractéristique)**.** *Soit $\Theta = (\theta_k)_{k\geqslant 1}$ une suite de réels strictement positifs telle que $g_{\Theta} \in F(r, \gamma, K)$ où $r > 0$ et $\gamma > 0$. On a la convergence en loi, pour la topologie de la convergence uniforme locale dans $\mathbb{D}$*

$$p_n \xrightarrow[n\to\infty]{\text{loi}} F, \tag{9.2.10}$$

où

$$F(z) = \exp\left(-\sum_{k\geqslant 1} \frac{z^k}{k} X_k\right), \quad X_k = \sum_{\ell | k} \ell Y_\ell, \tag{9.2.11}$$

avec $(Y_\ell)_{\ell\geqslant 1}$ des variables aléatoires de Poisson indépendantes de paramètre $\frac{\theta_\ell}{\ell} r^\ell$.

Le théorème précédent donne en particulier la convergence du polynôme caractéristique pour les matrices de permutation d'Ewens. En effet, pour le paramètre $\Theta \equiv \theta$ constant, la fonction g_{Θ} est dans la classe $F(1, \theta, 0)$ et donc p_n converge vers une fonction limite comme conjecturé dans [CLZ24].

9.3 Questions ouvertes

9.3.1 Conditions de moment minimales et universalité

Comme conjecturé dans [BCG22], la convergence dans le Théorème 9.2.1 du polynôme caractéristique normalisé devrait être valable sous la condition minimale de moment

$$\mathbb{E}\left[|a_{12}a_{21}|^2\right] < \infty$$

sur les entrées $(a_{ij})_{i,j\geqslant 1}$, ce qui donne une condition de moment d'ordre quatre pour les matrices de Wigner et d'ordre deux pour les matrices de Girko. Le cadre adapté à cette conjecture est celui des matrices aléatoires elliptiques [NO15, Definition 1.3]. Ce modèle a été introduit par Girko dans [Gir86] et [Gir95] et consiste en les matrices suivantes. Considérons une famille $(a_{ij})_{i,j\geqslant 1}$ de variables aléatoires centrées et de carré intégrable, telle que $\{(a_{ij}, a_{ji}) : i < j\} \cup \{a_{ii} : i \geqslant 1\}$ soit une famille indépendante de variables aléatoires, et dont la loi est invariante par permutation des indices ou, de manière équivalente, la loi de (a_{ij}, a_{ji}) coïncide avec celle de $(a_{i'j'}, a_{j'i'})$ si $|\{i,j\}| = |\{i',j'\}|$. Pour

$$\mathbb{E}[|a_{12}|^2] = 1 \quad \text{and} \quad \mathbb{E}[a_{12}a_{21}] = t,$$

la matrice $A_n = (a_{ij})_{1\leqslant i,j\leqslant n}$ est dite t-Girko. La convergence de la mesure spectrale vers la distribution uniforme sur l'ellipse a été démontrée sous différentes conditions sur les variables, voir [NO15; OR14; Nau13]. Nous nous attendons à ce que la version suivante du Théorème 9.2.1 soit valable pour les matrices t-Girko générales décrites ci-dessus. En notant $\tau = \mathbb{E}[a_{12}^2]$, $s = \mathbb{E}[a_{11}^2] - t - \tau$ et $q = \mathbb{E}[(a_{12}a_{21} - t)^2] - t^2 - \tau^2$, la limite de $\det\left(1 + tz^2 - z\frac{A_{n,t}}{\sqrt{n}}\right)\exp(-ntz^2/2)$ devrait être donnée par :

$$\sqrt{1 - \tau z^2}\,\mathrm{e}^{-sz^2/2}\mathrm{e}^{-qz^4/4}\mathrm{e}^{-\sum_{k\geqslant 1} Y_k \frac{z^k}{\sqrt{k}}}$$

où $(Y_k)_{k\geqslant 1}$ sont des variables aléatoires gaussiennes complexes centrées telles que Y_1 a la même variance que a_{11}, Y_2 a la même variance que $a_{12}a_{21}$, et, pour $k \geqslant 3$, la variance de Y_k est la somme de la puissance k de la variance de a_{12} et de la puissance k de la covariance de a_{12} et a_{21}, c'est à dire, $\mathbb{E}[Y_k^2] = \mathbb{E}[a_{12}^2]^k + \mathbb{E}[a_{12}a_{21}]^k = \tau^k + t^k$ et $\mathbb{E}[|Y_k|^2] = \mathbb{E}[|a_{12}|^2]^k + \mathbb{E}[a_{12}\overline{a_{21}}]^k = 1 + \mathbb{E}[a_{12}\overline{a_{21}}]^k$.

9.3.2 Matrices à coefficients dans $\{0, 1\}$

Dans le travail [Cos23], une convergence du polynôme caractéristique a été obtenue pour des matrices dont les coefficients sont des variables de Bernoulli indépendantes avec espérance non nulle. La fonction holomorphe aléatoire limite peut être exprimée à l'aide de variables de Poisson. On pourrait s'interroger sur l'existence d'un analogue de l'ensemble de Ginibre elliptique pour de telles matrices, ainsi que sur la convergence de leur polynôme caractéristique.

9.3.3 Gaz de Coulomb déterminantaux

La convergence du Théorème 9.2.1 est une première étape vers la convergence du polynôme caractéristique en dehors du support de la mesure d'équilibre pour des matrices aléatoires elliptiques générales. Néanmoins, une autre approche aurait pu être suivie, consistant à considérer l'ensemble de Ginibre elliptique comme un cas particulier de gaz de Coulomb déterminantal. Dans cette perspective, il pourrait être possible de démontrer la convergence des traces en adaptant les résultats de [AHM15], et de montrer la tension du polynôme caractéristique en dehors du support de la mesure d'équilibre pour des gaz de Coulomb déterminantaux plus généraux en utilisant, par exemple, les résultats de [AC23].

9.3.4 Polynôme caractéristique à l'intérieur du support

Le Théorème 9.2.6 montre la convergence du polynôme caractéristique en dehors du support de la mesure spectrale limite. On peut s'interroger sur une étude similaire à l'intérieur de la région où se trouvent les valeurs propres, c'est-à-dire sur la limite de $\log p_n(z)$ pour z situé à l'intérieur du support limite. Le développement du logarithme donne

$$\log p_n(z) = \sum_{k=1}^{n} \log(1 - z\lambda_{k,n}) = n \int \log(1 - zu)\mu_n(\mathrm{d}u)$$

de sorte que l'analyse asymptotique peut être vue comme un théorème central limite pour la statistique logarithmique. Les limites des fluctuations de $\log |p_n(z)|$ pour les matrices de Ginibre à l'intérieur du disque unité ont été établies comme étant gaussiennes [RV07] et le champ limite dans le bulk est le champ libre gaussien. Comme suggéré par les résultats de Webb et Wong [WW19], l'échelle de fluctuations serait différente comparée à celle de la région extérieure. Des théorèmes centraux limites pour les statistiques linéaires ont été démontrés par Rider et Silverstein pour les matrices de Girko complexes générales [RS06] et par [CES21] dans le cas à entrées réelles, sous des hypothèses de régularité sur les fonctions test. Dans une autre direction universelle, des résultats de fluctuations ont été établis pour les statistiques linéaires des gaz de Coulomb [LS18; Bau+19], où le champ limite est également le champ libre gaussien.

9.3.5 Fluctuations des parties réelles

Une autre direction serait d'étudier les fluctuations des parties réelles des valeurs propres des matrices issues de l'ensemble de Ginibre elliptique. Pour des matrices du GUE, il est connu suite au travail de Gustavsson [Gus05] que la k-ième valeur propre présente des fluctuations gaussiennes autour de sa position attendue dans le semi-cercle, dans le bulk lorsque $\frac{k}{n} \to a \in (0, 1)$ et au bord du spectre lorsque $k \to \infty$ et $\frac{k}{n} \to 0$. La démonstration repose sur un résultat de Costin et Lebowitz [CL95] ainsi que de Soshnikov [Sos00b], qui établissent des fluctuations gaussiennes pour le nombre de points d'un processus ponctuel déterminantal situés dans un certain intervalle. Puisque les valeurs propres de l'ensemble de Ginibre elliptique forment un processus ponctuel déterminantal et que les résultats de [ADM23] donnent des asymptotiques pour le noyau associé, on pourrait chercher à obtenir des fluctuations gaussiennes pour leurs parties réelles en utilisant ces techniques.

Chapitre 10

Produits de matrices unitaires

Ce chapitre présente un problème connu sous le nom de problème de Horn multiplicatif, ou problème de Horn unitaire. Ce problème porte sur la caractérisation des valeurs propres d'un produit de matrices unitaires lorsque les spectres de chacun des facteurs sont fixés. La Section 10.1 introduit ce problème du point de vue de l'algèbre linéaire. La résolution du problème de Horn unitaire fait intervenir des coefficients combinatoires appelés coefficients de Littlewood-Richardson quantiques, comptant certaines courbes rationnelles, présentés en Section 10.1.2. Dans les Section 10.2.1 et 10.2.2, nous présentons nos résultats sur une version probabiliste du problème de Horn unitaire, correspondant respectivement aux articles [FT24] et [Fra24], présentés dans les Chapitres 6 et 7 de cette thèse.

10.1 Valeurs propres d'un produit de matrices unitaires

10.1.1 Problème matriciel

Le problème de Horn unitaire, ou multiplicatif, pose la question suivante :

Étant données deux matrices unitaires, quelles valeurs propres peut avoir leur produit ?

Cette question a été résolue par Agnihotri et Woodward [AW98], qui ont fourni des inégalités caractérisant les valeurs propres possibles d'un produit de matrices unitaires. Parallèlement, Belkale [Bel01] a abordé la même question en résolvant un problème de Katz [Kat96] concernant les systèmes locaux sur la sphère de Riemann. Biswas [Bis98] avait auparavant résolu le problème de Horn multiplicatif en dimension $n = 2$.

Soit $n \geqslant 1$ un entier, et soient $A, B \in \mathrm{U}(n)$ deux matrices unitaires. À multiplication près par $\det(A)$ et $\det(B)$, qui sont des nombres complexes de module 1, on peut supposer que A et B sont de déterminant unité. Ainsi, dans cette section, nous considérons les matrices du groupe spécial unitaire

$$\mathrm{SU}(n) \coloneqq \{U \in \mathcal{M}_n(\mathbb{C}) \mid U^*U = I_n,\ \det(U) = 1\}\,.$$

Pour $A \in \mathrm{SU}(n)$, ses valeurs propres sont sur le cercle unité $\mathbb{S}^1$ et peuvent être paramétrées par des angles

$$\alpha = (\alpha_1 \geqslant \cdots \geqslant \alpha_n)\ ,$$

où $\alpha_k \in [0, 1]$ pour $1 \leqslant k \leqslant n$ et tels que $\sum_{1 \leqslant k \leqslant n} \alpha_k \in \mathbb{N}$. Notons

$$\mathcal{O}(\alpha) \coloneqq \left\{U\,\mathrm{Diag}(\mathrm{e}^{2i\pi\alpha_1}, \ldots, \mathrm{e}^{2i\pi\alpha_n})U^*,\ U \in \mathrm{U}(n)\right\}$$

l'orbite de α, c'est-à-dire l'ensemble des matrices de $\mathrm{SU}(n)$ ayant pour valeurs propres $e^{2i\pi\alpha_1}, \dots, e^{2i\pi\alpha_n}$. Les résultats d'Agnihotri et Woodward [AW98], ainsi que de Belkale [Bel01], donnent des inégalités nécessaires et suffisantes reliant les valeurs propres des matrices (A, B, C) satisfaisant $ABC = I_n$. Introduisons l'espace de matrices correspondant à un nombre quelconque de facteurs $A_1, \dots, A_\ell$ avec $\ell \geqslant 1$ et d'orbites prescrites $(\theta_1, \dots, \theta_\ell)$ où $\theta_k = (\theta_{k,1} \geqslant \dots \geqslant \theta_{k,n})$:

$$\{(A_1, \dots, A_\ell) \in \mathcal{O}(\theta_1) \times \dots \times \mathcal{O}(\theta_\ell) \mid A_1 \cdots A_\ell = I_n\} \ .$$

L'ensemble précédent est stable par conjugaison de chaque facteur. Notons le quotient

$$\mathcal{M}(\theta_1, \dots, \theta_\ell) \coloneqq \{(A_1, \dots, A_\ell) \in \mathcal{O}(\theta_1) \times \dots \times \mathcal{O}(\theta_\ell) \mid A_1 \cdots A_\ell = I_n\} / \mathrm{SU}(n) \ . \quad (10.1.1)$$

Le problème de Horn multiplicatif demande alors :

pour quels triplets (α, β, γ), l'ensemble $\mathcal{M}(\alpha, \beta, \gamma)$ est-il non vide ?

10.1.2 Cohomologie quantique des Grassmanniennes

Nous décrivons dans cette section les résultats d'Agnihotri, Woodward [AW98] et de Belkale [Bel01] qui donnent une caractérisation des triplets de valeurs propres de produits de matrices unitaires. Cette caractérisation est exprimée à l'aide de coefficients combinatoires appelés coefficients de Littlewood-Richardson quantiques ou invariants de Gromov–Witten.

Coefficients de Littlewood-Richardson quantiques et inégalités

Pour $1 \leqslant k \leqslant n$, nous notons

$$\mathbb{G}r(k, n) \coloneqq \{V \in \mathbb{C}^n \mid \dim(V) = k\}$$

la Grassmannienne des espaces de dimension k de $\mathbb{C}^n$. Un *drapeau* $\mathcal{F}$ est une collection d'espaces vectoriels

$$\{0\} = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n$$

telle que $\dim(F_k) = k$ pour tout $0 \leqslant k \leqslant n$. De plus, pour $1 \leqslant k \leqslant n$, nous notons

$$\mathcal{P}_n^k \coloneqq \{I = (i_1, \dots, i_k) \mid 1 \leqslant i_1 < i_2 \dots < i_k \leqslant n\}$$

les k-uplets ordonnés d'éléments distincts de $\{1, \dots, n\}$.

Definition 10.1.1 (Variété de Schubert)**.** Soit $\mathcal{F}$ un drapeau et soit $I = (i_1, \dots, i_k) \in \mathcal{P}_n^k$ des indices distincts. L'ensemble

$$\Omega_I(\mathcal{F}) \coloneqq \{L \in \mathbb{G}r(k, n) \mid \forall\ 1 \leqslant j \leqslant k, \dim(L \cap F_{i_j}) \geqslant j\}$$

est appelé la *variété de Schubert* associée au drapeau $\mathcal{F}$ et à l'ensemble I.

Les invariants de Gromov–Witten, ou coefficients de Littlewood-Richardson quantiques, sont définis comme le nombre de courbes rationnelles passant par certaines variétés de Schubert. Nous renvoyons à [MS04] pour une construction détaillée de ces invariants.

Definition 10.1.2 (Invariants de Gromov–Witten)**.** Soit $(I_1, \dots, I_\ell) \in (\mathcal{P}_n^r)^\ell$ une famille de sous-ensembles d'indices de cardinal $r \leqslant n$, et soit $d \geqslant 0$ un entier. Soient $p_1, \dots, p_\ell$ des points de $\mathbb{P}^1_{\mathbb{C}}$ et $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ des drapeaux. L'*invariant de Gromov–Witten* $\langle \sigma_{I_1}, \dots, \sigma_{I_\ell} \rangle_d$ est le nombre d'applications holomorphes $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{G}r(r, n)$ de degré d telles que, pour tout $1 \leqslant k \leqslant \ell$, on ait $f(p_k) \in \Omega_{I_k}(\mathcal{F}_k)$. Ce nombre est défini comme étant nul s'il existe une infinité de telles applications.

Nous énonçons le théorème principal qui résout le problème de Horn multiplicatif, établi par Agnihotri et Woodward [AW98] ainsi que Belkale [Bel01].

Theorem 10.1.3 (Caractérisation des valeurs propres de produits de matrices unitaires, [AW98; Bel01])**.** *Soit $n \geqslant 1$ et soit $(\theta_1, \dots, \theta_\ell)$ une famille telle que pour tout $1 \leqslant k \leqslant \ell$, on ait :*

$$\theta_{k,1} \geqslant \cdots \geqslant \theta_{k,n}, \ \sum_{i=1}^{n} \theta_{k,i} = 0 \ \text{ et } \theta_{k,1} - \theta_{k,n} \leqslant 1.$$

Alors, il existe des matrices $A_1, \dots, A_\ell \in \mathrm{SU}(n)$ telles que pour chaque $1 \leqslant k \leqslant \ell$, $A_k \in \mathcal{O}(\theta_k)$ et $A_1 \cdots A_\ell = I_n$ si et seulement si les inégalités suivantes sont satisfaites :

$$\sum_{k=1}^{\ell} \sum_{i \in I_k} \theta_{k,i} \leqslant d.$$

pour tout choix de $I_1, \dots, I_\ell$ de taille $r < n$ tel que $\langle \sigma_{I_1}, \dots, \sigma_{I_\ell} \rangle_d > 0$.

Une liste réduite d'inégalités nécessaires et suffisantes est obtenue en remplaçant la condition $\langle \sigma_{I_1}, \dots, \sigma_{I_\ell} \rangle_d > 0$ par $\langle \sigma_{I_1}, \dots, \sigma_{I_\ell} \rangle_d = 1$, voir [Bel01]. Le Théorème 10.1.3 donne une caractérisation pour les produits ayant un nombre quelconque de facteurs.

10.2 Contributions

10.2.1 Densité de probabilité et volume

Dans cette section, nous présentons nos résultats sur une version probabiliste du problème de Horn unitaire introduit à la Section 10.1. Dans le travail [FT24], nous obtenons une expression de la densité de probabilité des valeurs propres d'un produit de matrices unitaires comme une somme de volumes de polytopes explicites.

L'ensemble des classes de conjugaison du groupe unitaire $\mathrm{U}(n)$ est homéomorphe au quotient $\mathcal{H} = (\mathbb{R}^n/\mathbb{Z}^n)/S_n$, où le groupe symétrique S_n agit sur $\mathbb{R}^n/\mathbb{Z}^n$ par permutation des coordonnées. Cet espace quotient est décrit par l'ensemble des suites décroissantes de $[0, 1[^n$. Pour $\theta = (\theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_n) \in \mathcal{H}$, notons par $\mathcal{O}(\theta)$ la classe de conjugaison correspondante

$$\mathcal{O}(\theta) := \left\{ U e^{2i\pi\theta} U^*, \, U \in \mathrm{U}(n) \right\}, \quad \text{où} \quad e^{2i\pi\theta} = \begin{pmatrix} e^{2i\pi\theta_1} & 0 & \dots & 0 \\ 0 & e^{2i\pi\theta_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & e^{2i\pi\theta_n} \end{pmatrix}.$$

La structure de groupe de $\mathrm{U}(n)$ induit un produit de convolution

$$* : \mathcal{M}_1(\mathcal{H}) \times \mathcal{M}_1(\mathcal{H}) \to \mathcal{M}_1(\mathcal{H})$$

sur l'espace des mesures de probabilité sur $\mathcal{H}$, tel que pour $\theta, \theta' \in \mathcal{H}$, $\delta_\theta * \delta_{\theta'}$ soit la loi de la variable aléatoire $p(U_\theta U_{\theta'})$, où U_θ (resp. $U_{\theta'}$) est une variable aléatoire suivant la loi uniforme dans $\mathcal{O}(\theta)$ (resp. $\mathcal{O}(\theta')$) et où $p : \mathrm{U}(n) \to \mathcal{H}$ est l'application qui associe à un élément de $\mathrm{U}(n)$ sa classe de conjugaison dans $\mathcal{H}$.
Notons $\mathcal{H}_{\mathrm{reg}} = \{\theta \in \mathcal{H} \mid \theta_1 > \theta_2 > \ldots > \theta_n\}$ l'ensemble des classes de conjugaison régulières de $\mathrm{U}(n)$, c'est-à-dire celles de dimension maximale dans $\mathrm{U}(n)$. Pour $\alpha, \beta \in \mathcal{H}_{\mathrm{reg}}$, $\delta_\alpha * \delta_\beta$ admet une densité $\mathrm{d}\mathbb{P}[\cdot \mid \alpha, \beta]$ par rapport à la mesure de Lebesgue sur

$$\left\{\gamma \in \mathcal{H} \mid \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i - \sum_{i=1}^n \gamma_i \in \mathbb{N}\right\}.$$

Le cone des ruches toriques $\mathcal{C}_g$

Le résultat principal de [FT24] est une formule pour $\mathrm{d}\mathbb{P}[\cdot \mid \alpha, \beta]$ exprimée en termes du volume des polytopes similaires au modèle de la ruche (hive) de Knutson et Tao [KT99]. Pour $0 \leqslant d \leqslant n$, on définit la *ruche torique* $R_{d,n}$ comme l'ensemble

$$R_{d,n} := \left\{(v_1, v_2) \in [\![0, n]\!]^2, d \leqslant v_1 + v_2 \leqslant n + d\right\},$$

qui peut être représenté comme un hexagone discret via l'application $(v_1, v_2) \mapsto v_1 + v_2 e^{i\pi/3}$, voir la Figure 3.11 pour un cas particulier et sa représentation hexagonale.

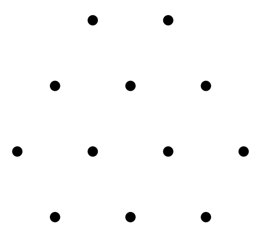

Figure 10.1: L'ensemble $R_{1,3}$ représenté via l'application $(v_1, v_2) \mapsto v_1 + v_2 e^{i\pi/3}$.

Frontière d'une ruche torique

Pour tout ensemble S et toute fonction $f : R_{d,n} \to S$, nous désignons par f^A (resp. f^B, f^C) le vecteur $(f((d-i) \vee 0, (n+d-i) \wedge n))_{0 \leqslant i \leqslant n}$ (resp. $(f(n+d-i \wedge n, i))_{0 \leqslant i \leqslant n}$, resp. $(f(n-i, i+d-n \vee 0))_{0 \leqslant i \leqslant n}$). Les vecteurs f^A, f^B et f^C correspondent respectivement aux frontières nord-ouest, est et sud-ouest de $R_{d,n}$ via la représentation hexagonale, voir la Figure 10.2.

Concavité losange torique

On appelle *losange* de $R_{d,n}$ toute suite $(v^1, v^2, v^3, v^4) \in (R_{d,n})^4$ correspondant à l'une des trois configurations de la Figure 10.3 dans la représentation hexagonale (dans laquelle $|v^i - v^{i+1}| = 1$ pour tout $1 \leqslant i \leqslant 3$).

Definition 10.2.1 (Étiquetage régulier)**.** Une fonction $g : R_{d,n} \to \mathbb{Z}_3$ est appelée *étiquetage régulier* lorsque

- $g_i^A = n + i[3]$, $g_i^B = i[3]$ et $g_i^C = i[3]$,

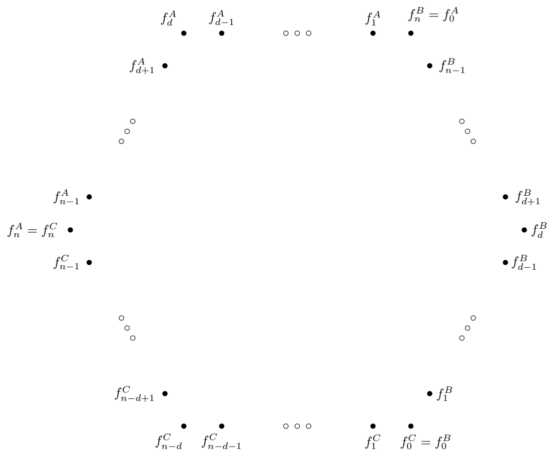

Figure 10.2: Les vecteurs des valeurs frontières f^A, f^B et f^C.

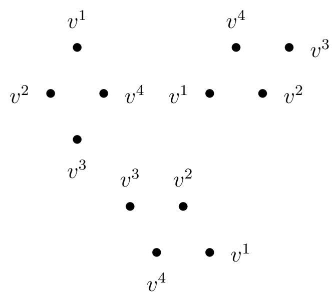

Figure 10.3: Les trois losanges possibles (v^1, v^2, v^3, v^4) (la position des sommets ne peut pas être permutée).

- pour tout losange $\ell = (v^1, v^2, v^3, v^4)$,

$$\big(g(v^2) = g(v^4)\big) \Rightarrow \{g(v^1), g(v^3)\} = \{g(v^2) + 1, g(v^2) + 2\}.$$

Un losange (v^1, v^2, v^3, v^4) pour lequel $(g(v^1), g(v^2), g(v^3), g(v^4)) = (a, a+1, a+2, a+1)$ pour un certain $a \in \{0, 1, 2\}$ est appelé *rigide*. Le *support* d'un étiquetage régulier $g : R_{d,n} \to \mathbb{Z}_3$ est le sous-ensemble $Supp(g) \subset R_{d,n}$ des sommets de $R_{d,n}$ qui ne sont pas le sommet v_4 d'un losange rigide (v^1, v^2, v^3, v^4).

Un exemple d'étiquetage régulier est présenté à la Figure 10.4.

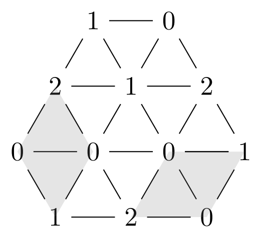

Figure 10.4: Un étiquetage régulier sur $R_{d,n}$. Les losanges rigides sont grisés.

Definition 10.2.2 (Cône de ruche torique)**.** Une fonction $f : R_{d,n} \to \mathbb{R}$ est dite *torique losanges concave* par rapport à un étiquetage régulier $g : R_{d,n} \to \mathbb{Z}_3$ lorsque $f(v_2)+f(v_4) \geqslant$

$f(v_1) + f(v_3)$ pour tout losange $\ell = (v^1, v^2, v^3, v^4)$, avec égalité si ℓ est rigide dans g.

Pour tout étiquetage régulier g, le *cône de ruche torique* $\mathcal{C}_g$ par rapport à g est le cône

$$\mathcal{C}_g = \left\{ f_{|Supp(g)} \mid f : R_{d,n} \to \mathbb{R} \text{ torique losanges concave par rapport à } g \right\}.$$

Le cône de ruche de Knutson et Tao [KT99] est alors un cas particulier du cône de ruche torique pour $d = 0$. Un exemple de fonction torique losange concave dans le cas $n = 3, d = 1$ est donné à la Figure 10.5.

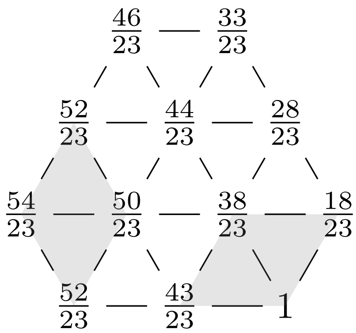

Figure 10.5: Une fonction torique losange concave pour $n = 3, d = 1$: les losanges grisés sont les losanges rigides donnant les cas d'égalité dans la concavité en losanges.

Definition 10.2.3 (Polytope $P^g_{\alpha,\beta,\gamma}$)**.** Soient $n \geqslant 3$ et $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ tels que $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ avec $d \in \mathbb{N}$. Soit g un étiquetage régulier sur $R_{d,n}$. Alors, $P^g_{\alpha,\beta,\gamma}$ désigne le polytope de $\mathbb{R}^{Supp(g) \setminus \partial R_{d,n}}$ constitué des fonctions dans $\mathcal{C}_g$ telles que

$$f^A = \left(\sum_{s=1}^n \beta_s + \sum_{s=1}^i \alpha_s \right)_{0 \leqslant i \leqslant n}, \; f^B = \left((d-i)^+ + \sum_{s=1}^i \beta_s \right)_{0 \leqslant i \leqslant n}, \; f^C = \left(d + \sum_{s=1}^i \gamma_s \right)_{0 \leqslant i \leqslant n}.$$

Un exemple d'élément de $P^g_{\alpha,\beta,\gamma}$ pour $n = 3$ et $d = 1$ est donné en Figure 10.5, pour $\alpha = \left(\frac{13}{23} \geqslant \frac{6}{23} \geqslant \frac{2}{23} \right)$, $\beta = \left(\frac{18}{23} \geqslant \frac{10}{23} \geqslant \frac{5}{23} \right)$ et $\gamma = \left(\frac{20}{23} \geqslant \frac{9}{23} \geqslant \frac{2}{23} \right)$.

Notre résultat principal est une formule pour la densité de la convolution de classes de conjugaison régulières, sous la forme d'une somme de volumes de polytopes de $\mathcal{C}_g$ associés à des étiquetages réguliers g.

Theorem 10.2.4 (Densité de probabilité pour le produit de classes de conjugaison)**.** *Soient $n \geqslant 3$ et $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ tels que $\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i = \sum_{i=1}^n \gamma_i + d$ avec $d \in \mathbb{N}$. Alors,*

$$\mathrm{d}\mathbb{P}[\gamma|\alpha,\beta] = \frac{(2\pi)^{(n-1)(n-2)/2} \prod_{k=1}^{n-1} k! \Delta'(\mathrm{e}^{2i\pi\gamma})}{n! \Delta'(\mathrm{e}^{2i\pi\alpha}) \Delta'(\mathrm{e}^{2i\pi\beta})} \sum_{g : R_{d,n} \to \mathbb{Z}_3 \text{ régulier}} \mathrm{Vol}_g(P^g_{\alpha,\beta,\gamma}), \qquad (10.2.1)$$

où $\Delta'(e^{2i\pi\theta}) = 2^{n(n-1)/2} \prod_{i<j} \sin(\pi(\theta_i - \theta_j))$ pour $\theta \in \mathcal{H}$, et Vol_g désigne le volume par rapport à la mesure de Lebesgue sur $\mathbb{R}^{Supp(g) \setminus \partial R_{d,n}}$.

Lien avec les connexions plates

Soit $\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)$ l'espace de modules des connexions plates à valeurs dans $\mathrm{SU}(n)$ sur la sphère à trois trous Σ_0^3, dont les holonomies autour des lacets a, b, c appartiennent respectivement à $\mathcal{O}(\alpha), \mathcal{O}(\beta)$ et $\mathcal{O}(\gamma)$. Nous renvoyons aux références [BM94; Ish99; Mor01] et [NS65], pour les définitions sur les connexions et leurs espaces de modules. Cet espace est relié au produit de matrices unitaires par l'isomorphisme

$$\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma) \simeq \{(U_1, U_2, U_3) \in \mathcal{O}(\alpha) \times \mathcal{O}(\beta) \times \mathcal{O}(\gamma),\ U_1U_2U_3 = Id_{\mathrm{SU}(n)}\}/\mathrm{SU}(n).$$

On obtient comme corollaire du Théorème 10.2.4, une expression du volume de l'espace de modules des connexions plates $\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)$ comme somme de volumes de polytopes explicites.

Corollary 10.2.5 (Volume des connexions plates à valeurs dans $\mathrm{SU}(n)$ sur la sphère)**.** *Soit $n \geqslant 3$ et considérons la forme volume canonique sur* $\mathrm{SU}(n)$*. Pour $\alpha, \beta, \gamma \in \mathcal{H}_{reg}$ tels que $|\alpha|_1,\ |\beta|_1,\ |\gamma|_1 \in \mathbb{N}$, alors* $\mathrm{Vol}\left[\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)\right] \neq 0$ *uniquement si*

$$\sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i + \sum_{i=1}^n \gamma_i = n + d \quad \textit{pour un certain } d \in \mathbb{N},$$

et dans ce cas :

$$\mathrm{Vol}\left[\mathcal{M}(\Sigma_0^3, \alpha, \beta, \gamma)\right] = \frac{2^{(n+1)[2]}(2\pi)^{(n-1)(n-2)}}{n!\Delta'(\mathrm{e}^{2i\pi\gamma})\Delta'(\mathrm{e}^{2i\pi\alpha})\Delta'(\mathrm{e}^{2i\pi\beta})} \sum_{g : R_{d,n} \to \mathbb{Z}_3 \textit{ régulier}} \mathrm{Vol}_g(P^g_{\alpha,\beta,\widetilde{\gamma}}),$$

où $\widetilde{\gamma} = (1-\gamma_n, \dots, 1-\gamma_1)$ et où les polytopes $P^g_{\alpha,\beta,\widetilde{\gamma}}$ sont définis dans la Définition 10.2.3.

10.2.2 Énumération des croisements dans les puzzles à deux étapes

Dans cette section, nous présentons nos résultats correspondant à l'article [Fra24], qui fait l'objet du Chapitre 7. Le résultat principal est le Théorème 10.2.10, qui dénombre des configurations dans une généralisation des étiquetages réguliers de la Définition 10.2.1.

Definition 10.2.6 (Réseau triangulaire)**.** Soit $n \geqslant 1$ et soit $\xi = \mathrm{e}^{\frac{i\pi}{3}}$. On note $T_n = \{r + s\xi, 0 \leqslant r + s \leqslant n\}$ l'ensemble des sommets du réseau triangulaire de taille n, et $E_n = \{(x, x + v) \mid x, x + v \in T_n \text{ et } v \in \{-\xi^{2l}, 0 \leqslant l \leqslant 2\}\}$ l'ensemble des arêtes de T_n. Les faces du réseau T_n sont des triangles que l'on dit directs (respectivement inversés) si les sommets correspondants $(x_1, x_2, x_3) \in T_n^3$ peuvent être étiquetés de sorte que $x_2 - x_1 = (1, 0)$ et $x_3 - x_1 = \xi$ (respectivement $x_3 - x_1 = \overline{\xi}$).

Les arêtes de E_n ne peuvent avoir que trois orientations possibles. Si $x = r + s\xi \in T_n$, on définit trois coordonnées (x_0, x_1, x_2) par

$$x_0 := n - (r + s),\ x_1 := r \text{ et } x_2 := s.$$

Definition 10.2.7 (Coordonnées et type d'une arête)**.** On dit qu'une arête $e = (x, x + v)$ *est de type l* pour $l \in \{0, 1, 2\}$ lorsque $v = -\xi^{2l}$. L'*origine* de e est x et les coordonnées de e sont le triplet $(e_0, e_1, e_2) = (x_0, x_1, x_2)$. La *hauteur* de e de type l est définie par $h(e) = e_l$. On définit aussi les arêtes frontières de E_n par

$$\begin{aligned}
\partial_0^{(n)} &:= (((n - r + 1, 0), (n - r, 0)), 1 \leqslant r \leqslant n) \\
\partial_1^{(n)} &:= \Big((n\xi + (r-1)\overline{\xi}, (n\xi + r\overline{\xi})), 1 \leqslant r \leqslant n\Big) \\
\partial_2^{(n)} &:= \Big(((r-1)\overline{\xi}, r\overline{\xi}), 1 \leqslant r \leqslant n\Big).
\end{aligned}$$

Definition 10.2.8 (Application de couleur)**.** Soit $n \geqslant 1$. Une *application de couleur* est une application $C : E_n \to \{0, 1, 3, m\}$ telle que les couleurs des arêtes autour de chaque face triangulaire, dans le sens horaire, soient $(0, 0, 0)$, $(1, 1, 1)$, $(1, 0, 3)$ ou $(0, 1, m)$, à une permutation cyclique près.

Les valeurs d'une application de couleur C sur les arêtes frontières sont notées $\partial C = (\partial_0 C, \partial_1 C, \partial_2 C)$ et sont définies pour $l \in \{0, 1, 2\}$ par $\partial_l C = C_{|\partial_l^{(n)}}$. On dit que C satisfait la condition au bord $\partial = (\partial_0, \partial_1, \partial_2)$ si $\partial C = \partial$.
Alternativement, on peut voir une application de couleur C comme un pavage de T_n par l'ensemble de tuiles étiquetées par leurs arêtes, donné dans la Figure 10.6, où les tuiles peuvent être tournées. Les deux dernières tuiles sont appelées respectivement lozanges 3 et m, en accord avec la couleur de leur arête centrale.

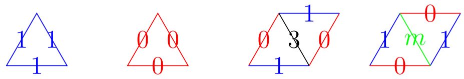

Figure 10.6: Tuiles possibles pour les applications de couleur

Nous considérerons des conditions au bord $\partial C \in \{0, 1\}^{3n}$ ayant un nombre égal d'arêtes de couleurs 0 et 1, respectivement notés n_0 et n_1, de sorte que $n_0 + n_1 = n$, voir la Figure 10.7 ci-dessous. De telles conditions au bord correspondent à celles des puzzles à deux étapes dans [Buc+16], où l'on a retiré les étiquettes 2 sur le bord. Ces puzzles fournissent une expression combinatoire pour les invariants de Gromov–Witten, voir [BKT03].

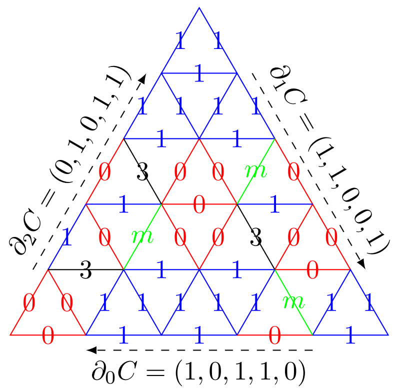

Figure 10.7: Une application de couleur sur E_5 avec condition au bord $\partial C = ((1, 0, 1, 1, 0), (1, 1, 0, 0, 1), (0, 1, 0, 1, 1))$.

Definition 10.2.9 (Nombres de gash)**.** Soit $C : E_n \to \{0, 1, 3, m\}$ une application de couleur. Pour tout $l \in \{0, 1, 2\}$ et toute arête $e \in \partial_l^{(n)}$, on note $n(C, e) = |\{e' \in \partial_l^{(n)} : h(e') < h(e) \text{ et } C(e') = 1\}|$ le nombre d'arêtes colorées en 1 situées à l'est (respectivement au nord, au sud) de e si $e \in \partial_0^{(n)}$ (respectivement $e \in \partial_1^{(n)}$, $e \in \partial_2^{(n)}$). Les *nombres de gash* de l'application de couleur C sont définis pour $l \in \{0, 1, 2\}$ par

$$G(C, l) \coloneqq \sum_{e \in \partial_l^{(n)} : C(e) = 0} n(C, e). \tag{10.2.2}$$

Par exemple, pour l'application de couleur C de la Figure 10.7, on a $G(C, 0) = 4$, $G(C, 1) = 4$, $G(C, 2) = 1$. Le résultat principal de notre travail [Fra24] est le Théorème 10.2.10

qui donne une formule pour le nombre d'arêtes 3 et m dans les applications de couleur, dépendant uniquement des nombres de gash. Le nombre d'arêtes m dans les applications de couleur est le nombre de losanges rigides dans l'étiquetage régulier correspondant, lequel encode les conditions d'égalité dans les polytopes apparaissant dans le Théorème 10.2.4. Nous renvoyons au Chapitre 7 pour plus de détails sur cette correspondance.

Theorem 10.2.10 (Nombre d'étiquettes dans les applications de couleur)**.** *Soit C une application de couleur sur E_n ayant n_0, respectivement n_1, arêtes de couleur 0, respectivement 1, sur chacun de ses bords. Soient $m(C)$ et $s(C)$ respectivement le nombre d'arêtes m et 3 dans C. Alors,*

$$m(C) = G(C,0) + G(C,1) + G(C,2) - n_0 n_1 \tag{10.2.3}$$

et

$$s(C) = 2n_0 n_1 - G(C,0) - G(C,1) - G(C,2)\ . \tag{10.2.4}$$

10.3 Questions ouvertes

10.3.1 Volumes des connexions plates sur des surfaces générales

Le Corollaire 10.2.5 fournit une formule pour le volume des connexions plates $\mathrm{SU}(n)$ sur la sphère à trois trous. De tels volumes sont liés à la mesure de Yang-Mills sur les surfaces de Riemann dans la limite des petites surfaces [For93], et il a été démontré dans [Wit92; MW99] que leur calcul pour des surfaces de Riemann compactes quelconques peut être réduit au cas de la sphère à trois trous par recollement. Une procédure inductive similaire est utilisée dans [Mir07] pour réduire le problème de volume de l'espace des modules de courbes au cas de genre zéro. Comprendre comment l'expression du Corollaire 10.2.5 se comporte lorsque l'on recolle des sphères à trois trous pour former des surfaces plus générales pourrait mener à des formules de volumes de connexions plates sur des surfaces de Riemann compactes.

10.3.2 Un modèle de ruche à deux étapes

Le travail de Knutson et Tao [KT99; KT03] a conduit à une description des coefficients de Littlewood-Richardson comme le nombre de points entiers dans un polytope, le polytope ruche. En plus de la formulation en puzzles pour les constantes de structure de la cohomologie de la variété de drapeaux à deux étapes de [Buc+16], on pourrait chercher une description de ces constantes de structure comme le nombre de points entiers dans un polytope qui généraliserait le polytope ruche de Knutson et Tao.

10.3.3 Extension à d'autres groupes de Lie

Pour la version hermitienne du problème de Horn, qui consiste à caractériser les valeurs propre d'une somme de matrices hermitiennes, des extensions aux matrices symétriques réelles et hermitiennes quaternioniques ont été faites, voir par exemple [Ful00] et [CMZ19]. D'autres extensions ont également été traitées dans [Par23] pour les groupes de Lie réductifs non compacts et dans [CM23] pour les groupes de Lie compacts. On peut poser la question d'extensions similaires pour la version multiplicative dans d'autres groupes de Lie G distincts du groupe unitaire $\mathrm{U}(n)$. En particulier, on pourrait chercher à exprimer la densité pour un produit de matrices de permutation, ou de matrices orthogonales. Le cas des matrices de $\mathrm{GL}(n)$ a été étudié dans [KO24].

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RÉSUMÉ

Cette thèse explore certains aspects de la solvabilité exacte en théorie des matrices aléatoires. Elle est structurée en deux parties principales.

La première partie traite d'un problème en grande dimension, sur comportement asymptotique des polynômes caractéristiques de matrices aléatoires. Nous nous concentrons sur deux modèles intégrables. Le premier est l'Ensemble de Ginibre Elliptique, une interpolation gaussienne entre l'Ensemble de Ginibre et son homologue hermitien, l'Ensemble Unitaire Gaussien. Le second modèle concerne les matrices de permutation, où la permutation sous-jacente suit la distribution d'Ewens généralisée pour laquelle la mesure d'une permutation dépend uniquement de sa structure en cycles. Pour ces deux modèles, nous établissons la convergence en loi du polynôme caractéristique vers une fonction analytique aléatoire lorsque la dimension des matrices tend vers l'infini. Cette convergence a lieu en dehors du support des valeurs propres et est complémentaire de la convergence des distributions spectrales.

La seconde partie concerne un problème en dimension fixée. Nous considérons des produits de matrices unitaires uniformément distribuées sur des orbites de conjugaison. Nous déterminons la densité de probabilité des valeurs propres de ce produit. Cette densité est liée au volume de l'espace des modules des connexions plates sur une sphère à trois trous. Notre formule fournit une expression positive pour la densité et pour ce volume sous la forme d'une somme de volumes de polytopes explicites. Ces polytopes émergent d'objets combinatoires appelés puzzles, permettant de calculer les coefficients d'intersection pour la cohomologie des variétés de drapeaux à deux sous-espaces. Nous explorons également certaines propriétés de ces puzzles.

MOTS CLÉS

Spectre de matrices aléatoires, fonctions analytiques aléatoires, cohomologie quantique, pavages du réseau triangulaire.

ABSTRACT

This thesis explores certain aspects of exact solvability in random matrix theory. It is structured into two main parts.

The first part examines a high dimensional problem on the asymptotic behavior of characteristic polynomials of random matrices. We focus on two integrable models. The first is the elliptic Ginibre Ensemble, a Gaussian interpolation between the Ginibre Ensemble and their Hermitian counterpart, the Gaussian Unitary Ensemble. The second model involves permutation matrices, where the underlying permutation follows the generalized Ewens distribution, for which the measure of a permutation only depends on its cycle structure. For both models, we establish the convergence in law of the characteristic polynomial, as the matrix dimension tends to infinity, towards a random analytic function. This convergence occurs outside the eigenvalue support and is complementary to the convergence of spectral distributions.

The second part is a fixed dimensional problem. We consider a product of unitary matrices that are uniformly distributed on fixed conjugacy orbits. We derive the probability density for the eigenvalues of this product. This probabilty density is related to the volume of moduli space of flat connections on the three-holed sphere. Our formula provides an positive expression for both the density and this volume as a sum of volumes of explicit polytopes. These polytopes arise from combinatorial objects called puzzles, which compute intersection coefficients for the cohomology of two-step flag varieties. We further investigate some properties of these puzzles.

KEYWORDS

Spectrum of random matrices, random analytic functions, quantum cohomology, tilings of the triangular lattice.