CHARACTERISTIC POLYNOMIAL OF GENERALIZED EWENS RANDOM PERMUTATIONS

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Abstract. We prove the convergence of the characteristic polynomial for random permutation matrices sampled from the generalized Ewens distribution. Under this distribution, the measure of a given permutation depends only on its cycle structure with weights assigned to each cycle length. The proof is based on uniform control of the characteristic polynomial using results from the singularity analysis of generating functions, together with the convergence of traces to explicit random variables expressed via a Poisson family. The limit function is the exponential of a Poisson series which has already appeared in the case of uniform permutation matrices. It is the Poisson analog of the Gaussian Holomorphic Chaos, related to the limit of characteristic polynomials for other matrix models such as Circular Ensembles, i.i.d. matrices, and Gaussian elliptic matrices.

1. Introduction

The study of the characteristic polynomial for random matrices has gained importance as it allows one to derive results on the behaviour of the corresponding eigenvalues. Coefficients of characteristic polynomials exhibit some combinatorial structure as shown by Diaconis and Gamburd [DG04] in the case of random unitary matrices sampled from the Haar measure. Instead of individual coefficients, one can consider the characteristic polynomial itself as a random variable in the space of analytic functions. The behavior of the characteristic polynomial outside of the support of the limit eigenvalue distribution is of particular interest. One main motivation for the latter is the analysis of outliers with respect to the global behavior of eigenvalues which is given by the convergence of the empirical eigenvalue distribution. This approach was followed by Bordenave, Chafaï and García-Zelada [BCG22], proving the convergence of the characteristic polynomial of Girko matrices, that is, matrices with i.i.d. centered entries under a universal second order moment condition. This allowed them to prove a convergence of the spectral radius for such matrices to one which could not be obtained from the convergence of the eigenvalue measure to the uniform law on the unit disk. Their work was inspired by the results of Basak and Zeitouni [BZ20] who studied outliers for eigenvalues of Toeplitz matrices.

The limit function obtained for the characteristic polynomial of Girko matrices in [BCG22] involves the exponential of a Gaussian analytic function. Such an expression is an example of a Gaussian log-correlated field, see [NPS23] and reference therein. The corresponding random distribution was introduced as the holomorphic multiplicative chaos. It arises as the limit of the characteristic polynomial for Circular-β Ensembles [CN19] and its Fourier coefficients are related to the enumeration of combinatorial objects called magic squares. The holomorphic multiplicative chaos also appears as the limit of the characteristic polynomial of Gaussian elliptic matrices which interpolate between Ginibre and GUE matrices [FG23]. This form of the limit was proved to be universal in [BCG22] for Girko matrices and is conjectured to hold for a larger class of elliptic matrices interpolating between i.i.d. and hermitian models.

Coste [Cos23] considered the case of non-centered entries following Bernoulli distribution. The characteristic polynomial for such matrices in the sparse regime was shown to converge towards a random analytic function expressed as the exponential of a Poisson series. This form is the Poisson analog of the holomorphic multiplicative chaos and has connections to the enumeration of multiset partitions. The same function was also identified as the limit of characteristic polynomial for sums of random permutation matrices where the permutation follows the uniform distribution by Coste, Lambert and Zhu [CLZ24]. The authors raised the question of extending their results to other measures on the space of permutations notably to the Ewens measure

[Ewe72], a measure in which the weight of a permutation depends only on its cycle structure.

The goal of this paper is to answer the previous question on the convergence of the characteristic polynomial in the context of generalized Ewens distributed permutations, which encompasses the Ewens and thus uniform cases. The generalized Ewens distribution was introduced by Nikeghbali and Zeindler [NZ13] as a generalization of the classical Ewens distribution by assigning different weights to each cycle lengths. Following the results of Chhaibi, Najnudel and Nikeghbali [CNN17] on the characteristic polynomial of Haar unitary matrices, Bahier [Bah19] showed the convergence of the characteristic polynomial of Ewens permutation matrices at a microscopic scale around one and near irrational angles on the unit circle. Here, we consider the characteristic polynomial in a different regime namely inside the open unit disk where there are no eigenvalues.

For $n \geqslant 1$, we denote by S_n the group of permutations of $\{1, \dots, n\}$.

Definition 1.1 (Generalized Ewens measure, [NZ13])**.** Let $\Theta = (\theta_k)_{k\geqslant 1}$ be a sequence of positive real numbers. For $n \geqslant 1$, the *generalized Ewens measure* is the probability measure $\mathrm{d}\mathbb{P}_n^{\Theta}$ on S_n defined by

$$\mathrm{d}\mathbb{P}_n^{\Theta}[\sigma] = \frac{1}{n! h_n^{\Theta}} \prod_{k=1}^{n} \theta_k^{C_k(\sigma)} \tag{1.1}$$

where for a permutation $\sigma \in S_n$ and $k \geqslant 1$, $C_k(\sigma)$ is the number of cycles of σ with length k.

The Ewens measure corresponds to the case where the sequence Θ is constant equal to $\theta > 0$ in which case $h_n^{\Theta} = \binom{\theta+n-1}{n}$. The uniform measure on S_n corresponds to the Ewens distribution with parameter $\theta = 1$. From the sequence $\Theta = (\theta_k)_{k\geqslant 1}$, one defines as in [NZ13],

$$g_{\Theta}(z) = \sum_{k\geqslant 1} \frac{\theta_k}{k} z^k \text{ and } G_{\Theta}(z) = \exp(g_{\Theta}(z)) \tag{1.2}$$

as formal power series. For the Ewens measure of parameter θ, g_{Θ} and G_{Θ} are holomorphic in $\mathbb{D}$ with $g_{\Theta}(z) = -\theta \log(1-z)$ and $G_{\Theta}(z) = (1-z)^{-\theta}$. By [Hug+13, Lemma 2.6], one has

$$G_{\Theta}(z) = \sum_{n\geqslant 0} h_n^{\Theta} z^n,$$

where h_n^{Θ} are the constants in the definition of the generalized Ewens distribution (1.1).

In this paper, we consider characteristic polynomials of random matrices associated to random permutations sampled from the generalized Ewens distribution. Since permutations $\sigma \in S_n$ can be viewed as permutation matrices of size n, we say that A_n follows the generalized Ewens distribution if it is the matrix obtained from a permutation σ sampled from (1.1). The characteristic polynomial $p_n(z) = \det(1 - zA)$ of a permutation matrix A can be expressed as

$$p_n(z) = \prod_{k=1}^{n} (1 - z^k)^{C_k^{(n)}}, \tag{1.3}$$

where $C_k^{(n)}, 1 \leqslant k \leqslant n$ are the cycle lengths of the associated random permutation. Note that the eigenvalues of A_n are explicit and are given by roots of unity located on the unit circle. Our result aims at showing the convergence of $(p_n)_{n\geqslant 1}$ as a sequence of random holomorphic functions defined on the unit disk. As in [BCG22; FG23; Cos23] and [CLZ24], we consider the limit of the characteristic polynomial in the region outside of the eigenvalue support, namely $p_n(z) = z^n \det(z^{-1} - A_n)$ so that for $z \in \mathbb{D}$, z^{-1} lies outside of the unit circle and $\det(z^{-1} - A_n)$ does not vanish.

2. Main result

2.1. **Convergence of the characteristic polynomial.** For $n \geqslant 1$ and $\Theta = (\theta_k)_{k\geqslant 1}$ as above, we consider A_n the random matrix associated to a permutation σ sampled from (1.1). In this paper, we consider characteristic polynomial

$$p_n(z) = \det(1 - zA_n) \tag{2.1}$$

inside the unit disk $z \in \mathbb{D} = \{x \in \mathbb{C} : |x| < 1\}$. Let us denote by $\mathcal{H}(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$ endowed with the topology of convergence on compact subsets of $\mathbb{D}$. Our main result is the convergence of p_n as a random variable in $\mathcal{H}(\mathbb{D})$ in law towards a limit function $F \in \mathcal{H}(\mathbb{D})$. The above convergence holds for parameters Θ such that the generating series g_Θ satisfies some conditions that we now define as Definition 2.1 which is an adaptation of a definition given in Section 5.2.1 of [Hwa94]. One can also find it as Definition 2.9 in [Hug+13] or Definition 2.8 in [NZ13].

Definition 2.1 (Logarithmic class function)**.** A function g is said to be in $F(r, \gamma, K)$ for $r > 0$, $\gamma \geqslant 0$ and $K \in \mathbb{C}$ if

- There exists $R > r$ and $\phi \in (0, \pi/2)$ such that g is holomorphic in $\Delta(r, R, \phi) \setminus \{r\}$ where $\Delta(r, R, \phi) = \{z \in \mathbb{C} : |z| \leqslant R, |\arg(z - r)| \geqslant \phi\}$.
- As $z \to r$, $g(z) = -\gamma \log(1 - z/r) + K + O(z - r)$.

In the case of the Ewens measure of parameter θ, we have $g_\Theta(z) = -\theta \log(1 - z)$ so that $g_\Theta \in F(1, \theta, 0)$.

Our main result is Theorem 2.2 which gives the convergence of the characteristic polynomial towards a limit function for sequences Θ such that g satisfies the conditions of Definition 2.1.

Theorem 2.2 (Convergence of the characteristic polynomial)**.** *Let $\Theta = (\theta_k)_{k\geqslant 1}$ be a sequence of positive real numbers such that $g_\Theta \in F(r, \gamma, K)$. We have the convergence in law, for the topology of local uniform convergence of p_n as $n \to +\infty$:*

$$p_n \to F : z \mapsto \exp\left(-\sum_{k\geqslant 1} \frac{z^k}{k} X_k \right), \tag{2.2}$$

where

$$X_k = \sum_{\ell | k} \ell Y_\ell, \tag{2.3}$$

with $(Y_\ell)_{\ell\geqslant 1}$ independent Poisson random variables with parameter $\frac{\theta_\ell}{\ell} r^\ell$.

The previous theorem gives in particular the convergence of the characteristic polynomial for Ewens permutation matrices. Indeed, for constant θ, the function $g_\Theta \in F(1, \theta, 0)$ so that p_n converges towards the limit function as conjectured in [CLZ24].

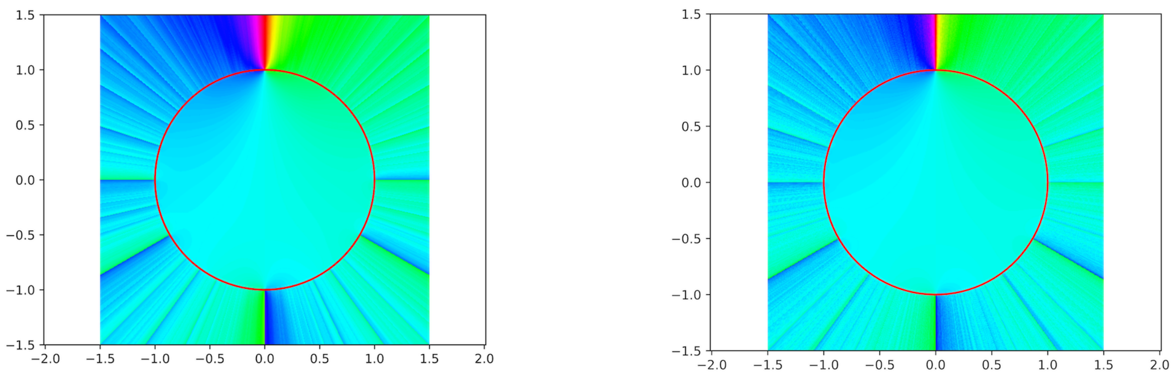

FIGURE 1. Phase portrait of p_n for an Ewens matrix of size $n = 10000$ with parameter $\theta = 100$ (left) and phase portrait of the limit function with same parameter (right). The unit circle is represented in red.

Remark 2.3 (Outside region)**.** Theorem 2.2 deals with the convergence in law for $z \in \mathbb{D}$ so that $p_n(z) = \det(1 - zA_n)$ does not vanish as eigenvalues of A_n are located on the unit circle. One can extend the previous to values of $p_n(z)$ for z in $\mathbb{C} \setminus \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| > 1\}$ under suitable normalization. Indeed, notice that the generalized Ewens distribution (1.1) is invariant under inversion, that is, if σ has distribution (1.1) then so does σ^{-1} as they both have the same cycle structure. Thus, $A_n = A_n^{-1}$ in law. Furthermore, for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$, $\det(1 - zA_n) = (-z)^n \det(A_n) \det(1 - z^{-1}A_n^{-1})$ so that if $\tilde{p}_n(z) = \frac{p_n(z)}{(-z)^n \det(A_n)}$ and $\iota(z) = \frac{1}{z}$, Theorem 2.2, gives the convergence in law on $\mathbb{C} \setminus \overline{\mathbb{D}}$ of $\tilde{p}_n$ to $F \circ \iota$.

2.2. **Method of proof.** The proof of Theorem 2.2 relies on the same structure as in [BCG22], which is recalled in Lemma 3.1. It is a consequence of the general fact stated in [Shi12] that a tight sequence of holomorphic functions in $\mathcal{H}(\mathbb{D})$ whose coefficients convergence in law for finite dimensionnal distributions converges to a random analytic function. We first show that the sequence $(p_n)_{n\geqslant 1}$ is tight which is Theorem 3.2 proved in section 4. The question of tightness for the Ewens model was raised in [CLZ24]. Here, tightness is archieved by a uniform control of the second moment of p_n. This control relies on results from Hwang [Hwa94] on singularity analysis for generating functions. The finite dimensionnal convergence of coefficients is obtained by showing the convergence of traces of powers, see the discussion above Theorem 3.3. The convergence of traces for generalized Ewens matrices was done in [Hug+13] and [NZ13]. We recall their results in Section 5 where Theorem 3.3 is proved. From these two results, one is able to derive the convergence of p_n towards a random analytic function F. The fact that F coincides with the exponential of a Poisson series is the purpose of Theorem 3.4 proved in section 6. In the rest of the paper, we assume that Θ is fixed and we write g and G for the functions defined in (1.2) for notation convenience.

3. Proof of Theorem 2.2

Recall that $\mathcal{H}(\mathbb{D})$ denotes the space of analytic functions on $\mathbb{D}$ endowed with the topology of local uniform convergence. In order to show the convergence in law of a sequence $(f_n)_{n\geqslant 1}$ in $\mathcal{H}(\mathbb{D})$, we rely on Lemma 3.1 which is close to Proposition 2.5 in [Shi12]. It is also stated as Lemma 3.2 in [BCG22] and proved therein.

Lemma 3.1 (Tightness and convergence of coefficients imply convergence of functions)**.** *Let $\{f_n\}_{n\geqslant 1}$ be a sequence of random elements in $\mathcal{H}(\mathbb{D})$ and denote the coefficients of f_n by $(\xi_k^{(n)})_{k\geqslant 0}$ so that for all $z \in \mathbb{D}$, $f_n(z) = \sum_{k\geqslant 0} \xi_k^{(n)} z^k$. Suppose also that the following conditions hold.*

(a) *The sequence $\{f_n\}_{n\geqslant 1}$ is a tight sequence of random elements of $\mathcal{H}(\mathbb{D})$.*
(b) *There exists a sequence $(\xi_k)_{k\geqslant 0}$ of random variables such that, for every $m \geqslant 0$, the vector $(\xi_0^{(n)}, \dots, \xi_m^{(n)})$ converges in law as $n \to \infty$ to $(\xi_0, \dots, \xi_m)$.*

Then, $f(z) = \sum_{k\geqslant 0} \xi_k z^k$ is a well-defined function in $\mathcal{H}(\mathbb{D})$ and f_n converges in law towards f in $\mathcal{H}(\mathbb{D})$ for the topology of local uniform convergence.

We thus need to show that the sequence $(p_n)_{n\geqslant 1}$ is tight and then study the limit of finite dimensionnal distributions for its coefficients. The first part is given by Theorem 3.2 which is proved in section 4.

Theorem 3.2 (Tightness)**.** *The sequence $(p_n)_{n\geqslant 1}$ is tight in $\mathcal{H}(\mathbb{D})$.*

It remains to study the coefficients of p_n. Let us write

$$p_n(z) = 1 + \sum_{k=1}^{n} (-z)^k \Delta_k(A_n)$$

where $\Delta_k(A)$ is the coefficient of z^k in $\det(1 + zA)$. Coefficients $\Delta_k(A_n)$ can be expressed via $(\mathrm{Tr}[A_n^\ell], 1 \leqslant \ell \leqslant k)$ so that

$$\Delta_k(A_n) = \frac{1}{k!} P_k \left(\mathrm{Tr}[A_n^1], \dots, \mathrm{Tr}[A_n^k] \right)$$

where the polynomials P_k do not depend on n. In order to study the convergence in law of coefficients $(\Delta_1(A_n), \dots, \Delta_k(A_n))$, it suffices to study the convergence of traces $\left(\mathrm{Tr}[A_n^1], \dots, \mathrm{Tr}[A_n^k]\right)$ which is given by Theorem 3.3. Recall that r denotes the radius of convergence of g, see Definition 2.1.

Theorem 3.3 (Convergence of coefficients)**.** *Let $k \geqslant 1$. We have the convergence in law as $n \to \infty$,*

$$(\mathrm{Tr}[A_n], \dots, \mathrm{Tr}[A_n^k]) \to (X_1, \dots, X_k) \tag{3.1}$$

where

$$X_k = \sum_{\ell | k} \ell Y_\ell \tag{3.2}$$

with $(Y_\ell, \ell \geqslant 0)$ are independent Poisson random variables with parameter $\frac{\theta_d}{d} r^d$.

Thanks to Lemma 3.1, Theorem 3.2 and Theorem 3.3, we derive that p_n converges towards the random analytic function $F \in \mathcal{H}(\mathbb{D})$ given by

$$F(z) = 1 + \sum_{k \geqslant 1} \frac{(-z)^k}{k!} P_k(X_1, \dots, X_k).$$

To obtain the expression of Theorem 2.2, we rely on Theorem 3.4, proved in Section 6 which yields the desired expression and ends the proof of Theorem 2.2.

Theorem 3.4 (Poisson expression for F)**.** *For every $z \in \mathbb{D}$, one has almost surely,*

$$F(z) = \exp(-f(z)) \tag{3.3}$$

where $f(z) = \sum_{k \geqslant 1} \frac{X_k}{k} z^k$ and where $(X_k)_{k \geqslant 1}$ are defined as in Theorem 2.2.

4. Tightness: proof of Theorem 3.2

The goal of this section is to prove Theorem 3.2. We start by Lemma 4.1 which reduces the tightness of a sequence of a functions (f_n) to proving tightness of their local supremum. This Lemma corresponds to Proposition 2.5 of [Shi12].

Lemma 4.1 (Reduction to uniform control)**.** *Let $(f_n)_{n \geqslant 1}$ be a sequence of random elements of $\mathcal{H}(\mathbb{D})$. If for every compact $K \subset \mathbb{D}$, the sequence $(\sup_{z \in K} |f_n(z)|)_{n \geqslant 1}$ is tight, then $(f_n)_{n \geqslant 1}$ is tight.*

It therefore suffices to show that $(\sup_{z \in K} |f_n(z)|)_{n \geqslant 1}$ is tight. By subharmonicity of $|f_n(z)|^2$, this is equivalent to show that $(\sup_{z \in K} \mathbb{E}[|f_n(z)|^2])_{n \geqslant 1}$ is bounded, see for instance [Shi12, Lemma 2.6]. We will show this for the sequence $(p_n)_{n \geqslant 1}$ of characteristic polynomials by giving a uniform control of the second moment of p_n which is Proposition 4.2. This control comes from an asymptotic given in Corollary 3.8 of [Hug+13] where we explicit the fact that the error is uniform for z in compact subsets of $\mathbb{D}$.

Recall that the functions g and G are defined in (1.2) by $g = \sum_{k \geqslant 1} \frac{\theta_k}{k} z^k$ and $G(z) = \exp(g_\Theta(z))$ for $|z| < r$ where r is the radius of convergence of g.

Proposition 4.2 (Second moment control)**.** *Assume that $g \in F(r, \gamma, K)$. Let $\delta \in (0,1)$. Then, for $z \in \mathbb{D}_\delta$,*

$$\mathbb{E}[|p_n(z)|^2] = \frac{G(r|z|^2)}{G(rz)G(r\overline{z})} + O\left(\frac{1}{n}\right) \tag{4.1}$$

where the O term holds uniformly in $z \in \mathbb{D}_\delta$.

Proof. Let $\delta \in (0,1)$. For $z \in \mathbb{D}_\delta$, one has using Corollary 3.6 of [Hug+13]

$$\sum_{n \geqslant 0} t^n h_n \mathbb{E}[|p_n(z)|^2] = \exp(g(t)) S_z(t) \tag{4.2}$$

where h_n are the coefficients of (1.1) and

$$S_z(t) = \frac{G(t|z|^2)}{G(tz)G(t\overline{z})}. \tag{4.3}$$

We now apply the method of [Hwa94] to $\exp(g(t))S_z(t)$ as done in [Hwa94, Section 5.3.2] therein.

For every $z \in \mathbb{D}$, the function $t \mapsto G(zt)$ is analytic for $|t| \leqslant r + \epsilon_1$ for some $\epsilon_1 > 0$ such

that $\delta(r+\epsilon_1) < r$ since $G(u) = \exp(g(u))$ and since that g is analytic in $\mathbb{D}_r$. Therefore, for every $z \in \mathbb{D}$, $t \mapsto S_z(t)$ is analytic for $|t| \leqslant r + \epsilon_1$.
By assumption, g is analytic for $|t| \in \Delta(\epsilon_2, \phi)$ for some $\epsilon_2 > 0$ and $0 < \phi < \frac{\pi}{2}$. Set $R = r + \min(\epsilon_1, \epsilon_2)$ and $\xi > 0$ such that $r\mathrm{e}^{\xi} < R$. As in the proof of Theorem 12 in [Hwa94], we write

$$h_n \mathbb{E}[|p_n(z)|^2] = \frac{1}{2i\pi}\int_{\Gamma} \frac{\exp(g(t))S_z(t)}{t^{n+1}}dt + \frac{1}{2i\pi}\int_{\Gamma'} \frac{\exp(g(t))S_z(t)}{t^{n+1}}dt$$

where

$$\Gamma = \{t : |t-1| = r(\mathrm{e}^{\xi} - 1), |\arg(t-r)| \geqslant \phi\}$$
$$\Gamma' = \{t : |t| = r\mathrm{e}^{\xi}, |\arg(t-r)| \geqslant \phi\}.$$

For the second integral over Γ', we may use that

$$|S_z(t)| \leqslant \frac{\sup_{|u|\leqslant \delta^2 r\mathrm{e}^{\xi}} |G(u)|}{(\inf_{|u|\leqslant \delta r\mathrm{e}^{\xi}} |G(u)|)^2} = C \tag{4.4}$$

where C does not depend on z. The contribution of this integral is $O(r^{-n}\mathrm{e}^{-n\xi})$ as in [Hwa94] and where the O term is uniform in z. The asymptotic of the integral over Γ involves the function S_z only via $U_z(t)$ where

$$U_z(t) = h(t)S_z(r\mathrm{e}^{-t})$$

with h defined with the parameters relative to g only. The asymptotic in [Hwa94] relies on the asymptotic development $U_z(t) = S_z(r) + O(|t|)$. For our concerns, we check that the error term is uniform in z. We have

$$U_z(t) = h(t)(S_z(r) + O(|t|))$$

where the O is uniform in z since the constant can be taken as $\sup_{|t|\leqslant r} |S_z'(t)|$ which can be bounded uniformly with respect to z by bounding values of G and G' in $\mathbb{D}_{r\delta}$ in a similar fashion as in (4.4). Since $h(t) = (1 + O(|t|))$ does not depend on z, we derive that $U_z(t) = S_z(r) + O(|t|)$ uniformly in $z \in \mathbb{D}_\delta$. The rest of the proof of [Hwa94] applies so that one derives the same asymptotic (4.1) with an error tern uniform in $z \in \mathbb{D}_\delta$. □

From Proposition 4.2, one derives that $\mathbb{E}[|p_n(z)|^2]$ is bounded by a deterministic function of z that does not depend on n so that the sequence $(p_n, n \geqslant 1)$ is tight which ends the proof of Theorem 3.2.

5. Convergence of traces: proof of Theorem 3.3

The purpose of this section is to prove Theorem 3.3 on the finite dimensionnal convergence for traces of monomials $A_n^1, \dots, A_n^k$. The study of the convergence of traces for random permutation matrices following the generalized Ewens distribution has been done in [NZ13]. The convergence of finite dimensionnal distribution is a consequence of a functional equality on generating function stated as (5.1) below which is Theorem 3.1 of [NZ13]

$$\sum_{n\geqslant 0} h_n \mathbb{E}\left[\exp\left(i\sum_{m=1}^{b} s_m C_m^{(n)}\right)\right] t^n = \exp\left(\sum_{m=1}^{b} \frac{\theta_m}{m}(\mathrm{e}^{is_m} - 1)t^m\right) G(t). \tag{5.1}$$

From (5.1), using the result of [Hwa94] on singularity analysis for generating functions, one derives as done in [NZ13, Corollary 3.2], that for every $k \geqslant 1$,

$$(C_1^{(n)}, \dots, C_k^{(n)}) \to (Y_1, \dots, Y_k)$$

with $(Y_\ell)_{\ell\geqslant 1}$ independent Poisson random variables with parameter $\frac{\theta_\ell}{\ell} r^\ell$. Using that

$$\mathrm{Tr}[A_n^k] = \sum_{\ell | k} \ell C_\ell^{(n)}$$

yields the result of Theorem 3.3 by the Cramer-Wold theorem.

6. Poisson Expression: proof of Theorem 3.4

Let $f(z) = \sum_{k\geqslant 1} \frac{X_k}{k} z^k$ where $X_k = \sum_{\ell|k} \ell Y_\ell$ with $(Y_\ell)_{\ell\geqslant 1}$ independent Poisson random variables with parameters $(\frac{r^\ell\theta_\ell}{\ell})_{\ell\geqslant 1}$. Recall that r is the radius of convergence of the series $g(z) = \sum_{k\geqslant 1} \frac{\theta_k}{k} z^k$ so that $\frac{1}{r} = \limsup_k \theta_k^{\frac{1}{k}}$. We first show that f is a well-defined function on the open disk $\mathbb{D}$ in Proposition 6.1. Computation of convergence radius for Poisson series were done in [CLZ24] for independent Poisson variables $(Y'_\ell)_{\ell\geqslant 1}$ with parameters $(\frac{d^\ell}{\ell})_{\ell\geqslant 1}$. In particular, for $d > 1$, the radius of convergence of $f' = \sum_{k\geqslant 1} \frac{X'_k}{k} z^k$ with $X'_k = \sum_{\ell|k} \ell Y'_\ell$ is almost surely equal to $\frac{1}{d}$, see Theorem 2.7 in [CLZ24].

Proposition 6.1 (Radius of convergence for limit function)**.** *Almost surely, the radius of convergence of f is greater than 1.*

Proof. To find the radius of convergence of f, one must compute $\limsup(\frac{X_k}{k})^{\frac{1}{k}} = \limsup X_k^{\frac{1}{k}}$. Let $\epsilon > 0$. There exists ℓ_0 such that for $\ell \geqslant \ell_0$,

$$\left|\frac{1}{r} - \sup_{\ell\geqslant\ell_0} \theta_\ell^{\frac{1}{\ell}}\right| \leqslant \frac{\epsilon}{r}$$

so that for $\ell \geqslant \ell_0$,

$$r^\ell\theta_\ell \leqslant (1+\epsilon)^\ell.$$

Define on the same probability space sequences $(Y_\ell)_{\ell\geqslant 1}$ and $(Y'_\ell)_{\ell\geqslant 1}$ having respective parameters $(\frac{r^\ell\theta_\ell}{\ell})_{\ell\geqslant 1}$ and $(\frac{d^\ell}{\ell})_{\ell\geqslant 1}$, such that $Y_\ell \leqslant Y'_\ell$ almost surely for $\ell \geqslant \ell_0$. Then, almost surely,

$$X_k \leqslant X'_k + \sum_{\substack{\ell|k\\ \ell\leqslant\ell_0}} \ell(Y_\ell - Y'_\ell)$$

where $X'_k = \sum_{\ell|k} \ell Y'_\ell$. We have that $\sum_{\substack{\ell|k\\ \ell\leqslant\ell_0}} \ell(Y_\ell - Y'_\ell) \leqslant \sum_{\ell=1}^{\ell_0} \ell(Y_\ell - Y'_\ell) = c$ where c is a random constant that does not depend on k so that almost surely,

$$\limsup X_k^{\frac{1}{k}} \leqslant \limsup \left(X'_k + c\right)^{\frac{1}{k}} \leqslant \limsup(X'_k)^{\frac{1}{k}} = 1+\epsilon$$

where we have used that $X'_k + c \leqslant X'_k(1+|c|)$ for the second inequality and that the convergence radius of $\sum_{k\geqslant 1} \frac{X'_k}{k} z^k$ is almost surely $\frac{1}{1+\epsilon}$ using Theorem 2.7 of [CLZ24]. Therefore, we have that, for every $\epsilon > 0$, the convergence radius r_f of f satisfies

$$r_f \geqslant \frac{1}{1+\epsilon},$$

so that $r_f \geqslant 1$ almost surely. □

Since $F(0) = 1$ and that $F \in \mathcal{H}(\mathbb{D})$, one can consider $\log(F)$ which is a well-defined analytic function in a neighborhood of the origin, where log is the principal branch of the logarithm. This function coincides with $-f$ so that they are both equal. Both functions are well-defined in the unit disk from which one derives the desired expression of Theorem 3.4.

7. Poisson Multiplicative Function

For the sake of completeness, we provide another representation for the limit function of Theorem 2.2. This representation given in Lemma 7.1 has the form of an infinite product and is inspired from [Cos23] where the exponential of a Poisson series appeared in the context of Bernoulli matrices.

Lemma 7.1 (Infinite product expression)**.** *For $z \in \mathbb{D}$, one has*

$$\exp(-f(z)) = \prod_{k\geqslant 1} (1-z^k)^{Y_k}. \tag{7.1}$$

Proof. The expression above is due to the inversion

$$\sum_{k\geqslant 1}\frac{X_k}{k}z^k=\sum_{\ell\geqslant 1}\ell Y_\ell\sum_{k\geqslant 1}\frac{z^{k\ell}}{k\ell}=-\sum_{\ell\geqslant 1}Y_\ell\log(1-z^\ell),$$

which can be performed since uniform convergence holds for $z\in\mathbb{D}$. □

As introduced in [CLZ24], the expression of $F=\exp(-f)$ is the Poisson analog of the Gaussian holomorphic chaos which was first introduced in [NPS23] for the study of the characteristic polynomial of matrices from Circular-β Ensembles and their coefficients. The Gaussian holomorphic chaos also appeared in limit expressions for characteristic polynomials of i.i.d. matrices [BCG22] and Gaussian elliptic matrices [FG23]. It is the Gaussian analog of F, replacing Poisson random variables by complex Gaussians. This provides an example of log-correlated field as correlations for such function r are given by $\mathbb{E}[r(z)\overline{r(w)}]=-\log(1-z\overline{w})$. For the generalized Ewens measure, the correlations are given by the generating function g as stated in Lemma 7.2.

Lemma 7.2 (Correlations of Poisson field)**.** *For $z,w\in\mathbb{D}$, one has*

$$Cov(f(z),f(w))=\sum_{a,b\geqslant 1}\frac{1}{ab}g(rz^a\overline{w}^b). \tag{7.2}$$

Proof. Since we want to compute correlations, we must consider the series

$$\sum_{k\geqslant 1}\frac{X_k-\mathbb{E}[X_k]}{k}z^k. \tag{7.3}$$

From Proposition 6.1, we know that the convergence radius of $\sum_{k\geqslant 1}\frac{X_k}{k}z^k$ is at least 1. Let us check that the same holds for $\sum_{k\geqslant 1}\frac{\mathbb{E}[X_k]}{k}z^k$ so that (7.3) is well-defined for $z\in\mathbb{D}$. Let $\epsilon>0$. As in Proposition 6.1, there exists $\ell_0\geqslant 1$ such that for $\ell\geqslant\ell_0:\theta_\ell r^\ell\leqslant(1+\epsilon)^\ell$. Thus,

$$\mathbb{E}[X_k]=\sum_{\ell|k}r^\ell\theta_\ell\leqslant\sum_{\substack{\ell|k\\ \ell\leqslant\ell_0}}(r^\ell\theta_\ell-(1+\epsilon)^\ell)+\sum_{\ell|k}(1+\epsilon)^\ell$$

so that $|\mathbb{E}[X_k]|\leqslant(c+1)\tau_k$ where $\tau_k=\sum_{\ell|k}(1+\epsilon)^k$ and $c=\sum_{\ell\leqslant\ell_0}|\theta_\ell r^\ell-(1+\epsilon)^\ell|$. The latter implies that $\limsup_k|\mathbb{E}[X_k]|^{1/k}\leqslant\limsup\tau_k^{1/k}=1+\epsilon$ from which one derives that the convergence radius of $\sum_{k\geqslant 1}\frac{\mathbb{E}[X_k]}{k}z^k$ is greater than $\frac{1}{1+\epsilon}$. Since ϵ was arbitrary, the convergence radius is greater or equal to one so that (7.3) is well-defined for $z\in\mathbb{D}$. For $z,w\in\mathbb{D}$, we thus compute

$$\begin{aligned}
Cov(f(z),f(w))&=\sum_{k,h}\frac{z^k\overline{w}^h}{kh}\sum_{\substack{i|h\\ j|k}}ijCov(Y_i,Y_j)\\
&=\sum_{k,h}\frac{z^k\overline{w}^h}{kh}\sum_{\ell|k,\ell|h}\ell^2Var[Y_\ell]\\
&=\sum_{k,h}\frac{z^k\overline{w}^h}{kh}\sum_{\ell|k,\ell|h}\ell\theta_\ell r^\ell\\
&=\sum_{\ell\geqslant 1}\frac{\theta_\ell r^\ell}{\ell}\sum_{a,b\geqslant 1}\frac{z^{a\ell}\overline{w}^{b\ell}}{ab}\\
&=\sum_{a,b\geqslant 1}\frac{1}{ab}\sum_{\ell\geqslant 1}\frac{\theta_\ell}{\ell}(rz\overline{w})^\ell\\
&=\sum_{a,b\geqslant 1}\frac{1}{ab}g(rz^a\overline{w}^b).
\end{aligned}$$

□

Remark 7.3. In the case of uniform permutations [CLZ24] or even for Ewens random permutations, that is, $\theta_k = \theta$ for some $\theta > 0$, one has $r = 1$ and $g(z) = -\theta \log(1-z)$ so that

$$Cov(f(z), f(w)) = -\theta \sum_{a,b \geqslant 1} \frac{1}{ab} \log(1 - z^a \overline{w}^b)$$

which is the analog of the log-correlations obtained for the Gaussian holomorphic chaos. In general, the correlations for arbitrary sequences Θ are given by g. Moreover, the expectation of the limit can be expressed using G for any $z \in \mathbb{D}$,

$$\mathbb{E}\left[\prod_{k \geqslant 1} (1 - z^k)^{Y_k}\right] = \prod_{k \geqslant 1} \mathrm{e}^{-\theta_k \frac{r^k z^k}{k}} = \frac{1}{G(rz)}.$$

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